University of
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# Dvoretzky covering problem for general measures

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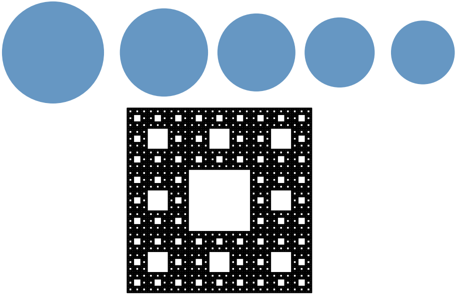
# Random covering sets



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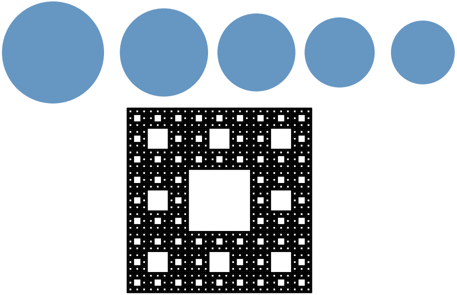

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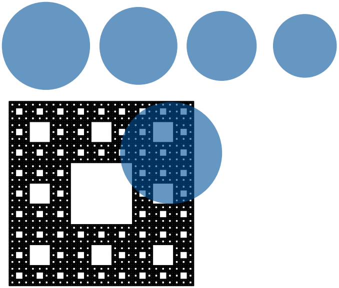

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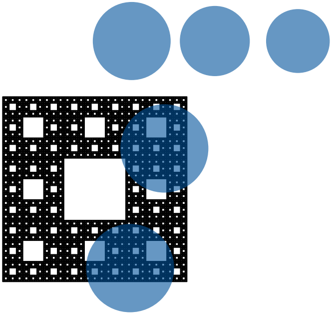

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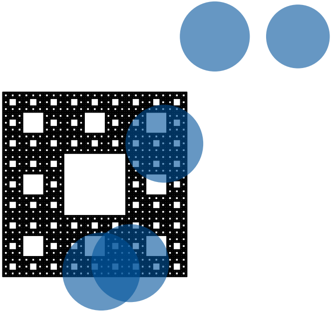

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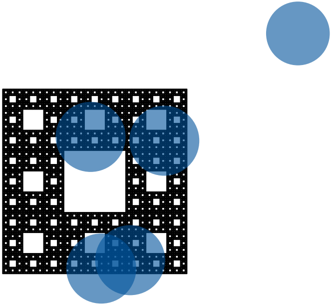

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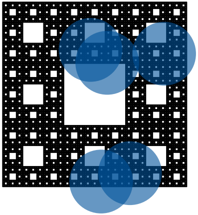

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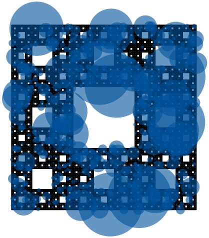

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# Random covering sets

**Definition**

The random covering set is the set $E_{\underline{r}} = E_{\underline{r}}(\omega)$ defined by

$$E_{\underline{r}} = \{x \in \mathbb{R}^d : x \in B(\omega_n, r_n) \text{ for infinitely many } n\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B(\omega_n, r_n).$$

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Remarks:

- Formally, $E_{\underline{r}}(\omega)$ is defined for every $\omega \in (\mathbb{R}^d)^{\mathbb{N}}$, and we are interested in the properties of $E_{\underline{r}}(\omega)$ for $\mathbb{P}_\mu = \mu^{\mathbb{N}}$-typical realisations of the process.

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- Assuming that $r_n$ is non-increasing does not result in loss of generality, since we may always achieve this by reordering the sequence (using independence of the centers).
- We assume that $r_n \downarrow 0$, to ensure that $E_{\underline{r}} \subset \operatorname{spt} \mu$, $\mathbb{P}_\mu$-almost surely.

# Random covering sets

For a number theorist it might be useful to think of random covering sets as a random analogue of $\psi$-well approximable sets defined for a positive function $\psi\colon \mathbb{N} \to \mathbb{R}$, by

$$\begin{aligned} W(\psi) &= \left\{ x \in \mathbb{R}^d \colon \left\| x - \frac{p}{q} \right\| \leqslant \frac{\psi(q)}{q} \text{ for infinitely many } p \in \mathbb{Z}^d,\, q \in \mathbb{N} \right\} \\ &= \left\{ x \in \mathbb{R}^d \colon x \in B\left( \frac{p}{q}, \frac{\psi(q)}{q} \right) \text{ for infinitely many } p \in \mathbb{Z}^d,\, q \in \mathbb{N} \right\} \end{aligned}$$

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**Question**

How large is $E_{\underline{r}}$ for a typical (w.r.t. $\mathbb{P}_\mu = \mu^{\mathbb{N}}$) realisation?

More specific questions include:

- Is $E_{\underline{r}} \neq \emptyset$ almost surely?
- What is $\mathcal{L}(E_{\underline{r}})$?
- Is $E_{\underline{r}} = \operatorname{spt} \mu$ almost surely?

# Dvoretzky covering problem

We are interested in the following generalisation of the last question:

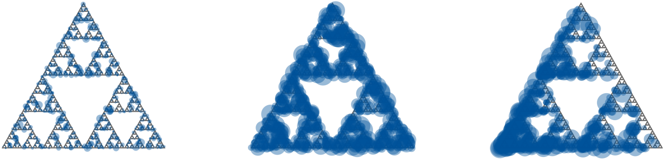
**Dvoretzky covering problem**

Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ and let $A \subset \mathbb{R}^d$ be measurable. When is $A \subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely?

Note: Always $\mathbb{P}_\mu(A \subset E_{\underline{r}}) \in \{0, 1\}$ by Kolmogorov's zero-one law.

The number theoretic analogue is, when is every point in a given set $A$ $\psi$-well approximable?

# Random covering sets



(a) $\underline{r} = (n^{-\frac{4}{5}})_{n=101}^{1100}$
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(c) $\underline{r} = (n^{-\frac{3}{5}})_{n=101}^{1100}$
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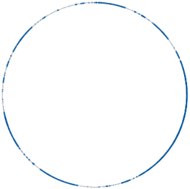
Figure: 1000 random balls on the Sierpinski triangle

More specifically, we want to find a characterisation for when $A \subset E_{\underline{r}}$, which only depends on $A$, $\mu$ and $\underline{r}$.

# Random covering sets: History



Classical case: $\mu = \mathcal{L}$, the Lebesgue measure on the one dimensional torus $\mathbb{T}$.

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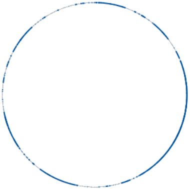

Classical case: $\mu = \mathcal{L}$, the Lebesgue measure on the one dimensional torus $\mathbb{T}$.

**Observation (Borel 1897)**

For a fixed $x \in \mathbb{T}$, we have

1. $x \notin E_{\underline{r}}$ almost surely, if $\sum_{n=1}^{\infty} r_n < \infty$,
2. $x \in E_{\underline{r}}$ almost surely, if $\sum_{n=1}^{\infty} r_n = \infty$.

Follows from the Borel-Cantelli lemma, since

$$\mathbb{P}(x \in B(\omega_n, r_n)) = \mathbb{P}(\omega_n \in B(x, r_n)) = \mathcal{L}(B(x, r_n)) = 2r_n,$$

and since $\sum_{n=1}^{\infty} 2r_n = \infty$ if and only if $\sum_{n=1}^{\infty} r_n = \infty$.

# Random covering sets: History

By Fubini's theorem, we have

$$\mathbb{E}(\mathcal{L}(E_{\underline{r}})) = \iint \chi_{E_{\underline{r}}}(x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P} = \iint \chi_{E_{\underline{r}}}(x)\,\mathrm{d}\mathbb{P}\,\mathrm{d}x = \int \mathbb{P}(x \in E_{\underline{r}})\,\mathrm{d}x,$$

so Borel's observation shows that, almost surely,

$$\mathcal{L}(E_{\underline{r}}) = \begin{cases} 0, & \text{if } \sum_n r_n < \infty \\ 1, & \text{if } \sum_n r_n = \infty. \end{cases}$$

For a number theorist, this is the random covering analogue of Khintchin's theorem, which states that

$$\mathcal{L}(W(\psi)) = \begin{cases} 0, & \text{if } \sum_q \psi(q) < \infty \\ 1, & \text{if } \sum_q \psi(q) = \infty. \end{cases}$$

This opens up natural follow up questions, i.e. what is $\dim_{\mathrm{H}} E_{\underline{r}}$ if $\sum_n r_n < \infty$?

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**Theorem (Dvoretzky 1956)**

1. *If $r_n \geqslant \frac{\log n}{n}$ for all large enough $n$, then $\mathbb{T} \subset E_{\underline{r}}$, almost surely.*

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Original Dvoretzky covering problem: What is the characterising condition?

# Dvoretzky covering problem: History

Borel's observation and Dvoretzky's result have the following corollary for polynomially decreasing sequences of radii $\underline{r} = (cn^{-t})_n$:

**Corollary**

*Let* $\underline{r} = (cn^{-t})_n$

1. *If* $t > 1$, *then* $\mathbb{T} \not\subset E_{\underline{r}}$ *almost surely.*
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Proof:

1. $\sum_n cn^{-t} < \infty$.
2. For all large enough $n \in \mathbb{N}$, $cn^{1-t} \geqslant \log n$, so

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$\square$

Interestingly, for the critical exponent $t = 1$, the covering property depends on the constant $c$.

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**Theorem**

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3. *(Erdös 1960, Mandelbrot 1972) If* $r_n = \frac{1}{2}n^{-1}$*, then* $\mathbb{T} \subset E_{\underline{r}}$ *almost surely.*

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The Dvoretzky covering problem for covering the full torus was solved by Shepp in 1972:

**Theorem (Shepp 1972)**

*If* $\underline{r} = (r_n)_n$*, then* $\mathbb{T}$ *is covered almost surely by* $E_{\underline{r}}$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(\sum_{k=1}^{n} 2r_k\right) = \infty.$$

This result was generalised by Kahane, who gave a potential theoretic characterisation for covering an arbitrary compact set $A \subset \mathbb{T}$.

# Capacity and coverings

Let $\Phi_{\underline{r}} \colon \mathbb{T}^2 \to \mathbb{R}$ denote the kernel function

$$\Phi_{\underline{r}}(x,y) = \exp\left(\sum_{n=1}^{\infty} \max\{2r_n - |x-y|, 0\}\right) = \exp\left(\sum_{n=1}^{\infty} \mathcal{L}(B(x,r_n) \cap B(y,r_n))\right).$$

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and the $\underline{r}$-capacity of a set $A \subset \mathbb{T}$ is defined by

$$\mathrm{Cap}_{\underline{r}}(A) = \sup\{I_{\underline{r}}(\nu)^{-1} \colon \nu \text{ is a Borel probability measure on } A\}.$$

Here we interpret $\infty^{-1} = 0$, so $\mathrm{Cap}_{\underline{r}}(A) = 0$ if and only if $I_{\underline{r}}(\nu) = \infty$ for all measures $\nu$ on $A$.

# Capacity and covering

Kahane proved the following:

**Theorem (Kahane 1990)**

*A compact set $C \subset \mathbb{T}$ is covered almost surely by $E_{\underline{r}}$ if and only if*

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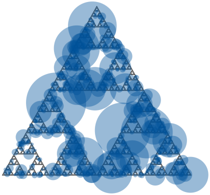
$$\mathrm{Cap}_{\underline{r}}(C) = 0.$$

- Note: One can show that $\mathrm{Cap}_{\underline{r}}(\mathbb{T}) = 0$ if and only if

$$I_{\underline{r}}(\mathcal{L}) = \iint \exp\left(\sum_{n=1}^{\infty} \max\{2r_n - |x-y|, 0\}\right) \mathrm{d}x\,\mathrm{d}y = \infty,$$

  i.e. the energy of the Lebesgue measure characterises the covering property.
- This is essentially how Shepp's proof works: He shows that $I_{\underline{r}}(\mathcal{L}) = \infty$ is equivalent with covering and that it is equivalent with the divergence of the series.

# Random covering sets: General setting



For random covering sets driven by arbitrary measures $\mu$ on $\mathbb{R}^d$, basic results work essentially in the same way as for the Lebesgue measure:

**Lemma**

*For any $x \in \mathbb{R}^d$, we have*

1. $x \notin E_{\underline{r}}$ $\mathbb{P}_\mu$*-almost surely, if* $\sum_{n=1}^{\infty} \mu(B(x, r_n)) < \infty$,
2. $x \in E_{\underline{r}}$ $\mathbb{P}_\mu$*-almost surely, if* $\sum_{n=1}^{\infty} \mu(B(x, r_n)) = \infty$.

- Again follows from Borel-Cantelli, since $\mathbb{P}_\mu(x \in B(\omega_n, r_n)) = \mu(B(x, r_n))$

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By using Fubini as earlier,

$$\mathbb{E}(\mu(E_{\underline{r}})) = \iint \chi_{E_{\underline{r}}}(x)\,\mathrm{d}\mu(x)\,\mathrm{d}\mathbb{P} = \iint \chi_{E_{\underline{r}}}(x)\,\mathrm{d}\mathbb{P}\,\mathrm{d}\mu(x) = \int \mathbb{P}(x \in E_{\underline{r}})\,\mathrm{d}\mu(x),$$

so

$$\mu(E_{\underline{r}}) = \begin{cases} 1, & \text{if } \sum_{n=1}^{\infty} \mu(B(x, r_n)) = \infty, \text{ for all } x \in \operatorname{spt}\mu, \\ 0, & \text{if } \sum_{n=1}^{\infty} \mu(B(x, r_n)) < \infty, \text{ for all } x \in \operatorname{spt}\mu. \end{cases}$$

Intermediate behaviour is possible.

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$$\mu(E_{\underline{r}}) = \begin{cases} 1, & \text{if } \sum_{n=1}^{\infty} \mu(B(x,r_n)) = \infty, \text{ for all } x \in \operatorname{spt}\mu, \\ 0, & \text{if } \sum_{n=1}^{\infty} \mu(B(x,r_n)) < \infty, \text{ for all } x \in \operatorname{spt}\mu. \end{cases}$$

Intermediate behaviour is possible.

In fact, we get something stronger: if $\nu$ is a Borel probability measure on $A$, and $\sum_n \mu(B(x,r_n)) = \infty$ for all $x \in A$, then

$$\mathbb{E}(\nu(E_{\underline{r}})) = \iint \chi_{E_{\underline{r}}}(x)\,\mathrm{d}\nu(x)\,\mathrm{d}\mathbb{P} = \int \mathbb{P}(x \in E_{\underline{r}})\,\mathrm{d}\nu(x) = 1,$$

so $\nu(E_{\underline{r}}) = 1$, almost surely.

# Random covering sets: General setting

Prior results for Hausdorff dimensions of random covering sets include

- $\mu = \mathcal{L}^d$ on $\mathbb{R}^d$; arbitrary $\underline{r}$ (Jaffard 2000, Fan–Wu 2004)
- $\mu = \mathrm{vol}^d$ on Riemannian manifold; arbitrary $\underline{r}$ (Feng–Järvenpää–Järvenpää–Suomala, 2018)
- $\mu =$ Gibbs measure on topological Markov shift (self-similar measure on a homogeneous self-similar set); $\underline{r} = n^{-t}$ (Seuret, 2018)
- $\mu$ arbitrary Borel probability measure on $\mathbb{R}^d$; $\underline{r} = n^{-t}$ (Ekström–Persson, 2018)
- $\mu$ arbitrary Borel probability measure on $\mathbb{R}^d$; arbitrary $\underline{r}$ (Järvenpää–Järvenpää–Myllyoja–Stenflo, 2024; Järvenpää–Myllyoja–Seuret, 2025)

For the Dvoretzky covering problem much less was known.

# Polynomially decreasing radii

For polynomially decreasing sequences, the critical exponent was known in the following cases.

**Theorem (Tang 2012)**

*Let $\mu$ be a fully supported Borel probability measure on $\mathbb{T}$, and let $r_n = (n^{-t})_n$.*

1. *If $t > (\sup_{x\in\mathbb{T}} \underline{\dim}_{\mathrm{loc}}(\mu, x))^{-1}$, then $\mathbb{T} \not\subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely.*
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**Theorem (Seuret 2018)**

*Let $\mu$ be a Gibbs measure (for a Hölder potential) on an irreducible topological Markov shift $\Sigma$, and let $r_n = (n^{-t})_n$.*

1. *If $t > (\max_{x\in\Sigma} \dim_{\mathrm{loc}}(\mu, x))^{-1}$, then $\Sigma \not\subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely.*
2. *If $t < (\max_{x\in\Sigma} \dim_{\mathrm{loc}}(\mu, x))^{-1}$, then $\Sigma \subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely.*

The critical constant at the critical exponent was not known for any singular measure.

# Dvoretzky covering problem: General setting

For measures other than $\mathcal{L}$, a full characterisation for the Dvoretzky covering problem was only known in the following cases:

- (Fan-Karagulyan 2021) For $\mu = f\,\mathrm{d}\mathcal{L}$, under some reqularity assumptions for $f$.
- (Kahane 1990) For $\mu =$ Lebesgue measure on $\mathbb{T}^d$, but for simplices (homothetic triangles in $\mathbb{R}^2$) instead of balls.

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In particular, the full characterisation was not known for any singular measure.

# Polynomially decreasing radii

Our first result is the following lemma:

**Lemma**

*Let $\mu$ be a Borel probability measure on $\mathbb{R}$ (or $\mathbb{T}$). Let $\varepsilon > 0$ and $A \subset \mathbb{R}$ be an analytic set, and assume that*

$$\sum_{n=1}^{\infty} \mu(B(x, r_n))^{1+\varepsilon} = \infty,$$

*for all $x \in A$. Then $A \subset E_{\underline{r}}$ almost surely.*

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- Proof is a quite elementary application of Borel-Cantelli.
- Recall that for $\varepsilon = 0$, the condition only implies $\nu(E_{\underline{r}}) = 1$ almost surely for all measures $\nu$ on $A$.

# Polynomially decreasing radii

The lemma has a straightforward corollary:

**Corollary (A.-Myllyoja 2026+)**

*Let* $\mu$ *be a Borel probability measure on* $\mathbb{R}^d$, $A \subset \mathbb{R}^d$ *be analytic, and* $\underline{r} = (n^{-t})_n$.

1. *If* $t > (\sup_{x \in A} \underline{\dim}_{\mathrm{loc}}(\mu, x))^{-1}$, *then* $A \not\subset E_{\underline{r}}$, $\mathbb{P}_\mu$*-almost surely.*
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2. *If $t < (\sup_{x \in A} \underline{\dim}_{\mathrm{loc}}(\mu, x))^{-1}$, then $A \subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely.*

- Our earlier lemma only works in $\mathbb{R}$ but a slightly weaker variant, which is enough for the corollary, works in $\mathbb{R}^d$.
- Again the critical case $t = (\sup_{x \in A} \underline{\dim}_{\mathrm{loc}}(\mu, x))^{-1}$ is much more subtle.

# Capacity and covering

For $\mathcal{L}$ define the kernel function $\Phi_{\underline{r}} \colon \mathbb{T}^2 \to \mathbb{R}$ by

$$\Phi_{\underline{r}}(x, y) = \exp\left(\sum_{n=1}^{\infty} \mathcal{L}(B(x, r_n) \cap B(y, r_n))\right),$$

and then the $\underline{r}$-energy of a measure $\nu$ by

$$I_{\underline{r}}(\nu) = \iint \Phi_{\underline{r}}(x, y) \, \mathrm{d}\nu(x) \, \mathrm{d}\nu(y),$$

and the $\underline{r}$-capacity of a set $A \subset \mathbb{T}$ by

$$\mathrm{Cap}_{\underline{r}}(A) = \sup\{I_{\underline{r}}(\nu)^{-1} \colon \nu \text{ is a Borel probability measure on } A\}.$$

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For $\mu$ define the kernel function $\Phi_{\mu,\underline{r}}\colon \mathbb{R}^{2d} \to \mathbb{R}$ by

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$$\mathrm{Cap}_{\mu,\underline{r}}(A) = \sup\{I_{\mu,\underline{r}}(\nu)^{-1} \colon \nu \text{ is a Borel probability measure with compact support on } A\}.$$

# Main result

For a Borel probability measure $\mu$ and a sequence of radii $\underline{r}$ we let

$$X_{\mu,\underline{r}} = \left\{ x \in \operatorname{spt}\mu \colon \sum_{n=1}^{\infty} \mu(B(x,r_n))^2 < \infty \right\}$$

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**Theorem (A.-Myllyoja 2026+)**

*Let $\mu$ be a Borel probability measure on $\mathbb{R}$, and let $A \subset \mathbb{R}$ be analytic. Then $A \subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely, if and only if*

$$\mathrm{Cap}_{\mu,\underline{r}}(A \cap X_{\mu,\underline{r}}) = 0.$$

- Note that for the Lebesgue measure on $\mathbb{T}$, the set $X_{\mu,\underline{r}}$ is either $\mathbb{T}$ or $\emptyset$, which is why it is not visible in Kahane's result.
- The condition depends only on $A$, $\underline{r}$ and $\mu$.
- Our earlier lemma shows that $A \setminus X_{\mu,\underline{r}} = \{x \in A \colon \sum_{n=1}^{\infty} \mu(B(x,r_n))^2 = \infty\}$ is covered almost surely automatically.

# Remarks

- Same result holds for measures on $\mathbb{T}$ instead of $\mathbb{R}$.
- By working in $\mathbb{R}$ instead of $\mathbb{T}$ we lose two properties which made earlier methods simpler: Compactness of the support of $\mu$ and rotational invariance.
- Even for $\mu = \mathcal{L}$ the result is new for analytic sets.

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- By working in $\mathbb{R}$ instead of $\mathbb{T}$ we lose two properties which made earlier methods simpler: Compactness of the support of $\mu$ and rotational invariance.
- Even for $\mu = \mathcal{L}$ the result is new for analytic sets.
- The major difficulty is in proving that $\mathrm{Cap}_{\mu,\underline{r}}(A \cap X_{\mu,\underline{r}}) = 0$ implies covering, the other direction is essentially well known.
- In general $I_{\mu,\underline{r}}(\mu) = \infty$ does not characterise covering of $\operatorname{spt} \mu$ (Counterexamples are given by Cantor measure and radii $cn^{-\frac{\log 3}{\log 2}}$, for some $c > 0$).
- As an application we can characterise the critical constant for the covering problem at the critical exponent for Hausdorff measures on strongly separated self-conformal sets.
- This characterisation depends on the multifractal structure of the average densities of the measure.

# Billard's condition

The following necessary condition for covering was known for arbitrary measures.

**Theorem (Billard 1965, Kahane 1985)**

*Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ and let $A \subset \mathbb{R}^d$ be compact, and assume that*

$$\sup_{x \in A} \sum_{n=1}^{\infty} \mu(B(x, r_n))^2 < \infty.$$

*If $\mathrm{Cap}_{\mu,\underline{r}}(A) > 0$, then $A \not\subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely.*

# Billard's condition: Proof

Let $A \subset \mathbb{R}^d$ and let $\nu$ be a Borel probability measure supported on $A$. Denote by $F_k = F_k(\omega) = A \setminus \bigcup_{n=1}^{k} B(\omega_n, r_n)$, and consider the random variable

$$M_{\nu,k}(\omega) = \int \frac{\chi_{F_k(\omega)}(x)}{\mathbb{P}(x \in F_k)} \,\mathrm{d}\nu(x).$$

This is easily seen to be a martingale with $\mathbb{E}(M_{\nu,x}) = 1$, hence it almost surely converges to some random variable $M_\nu$.

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This is easily seen to be a martingale with $\mathbb{E}(M_{\nu,x}) = 1$, hence it almost surely converges to some random variable $M_\nu$.

- If $M_\nu > 0$ with positive probability, then there is a positive probability that there exists $x \in A$, such that $x \notin \bigcup_{n=1}^{\infty} B(\omega_n, r_n)$.
- But if $x$ is not covered by any of the balls, then it is not covered infinitely often so (by the zero-one law) there is $x \notin E_{\underline{r}}$ almost surely.

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- But if $x$ is not covered by any of the balls, then it is not covered infinitely often so (by the zero-one law) there is $x \notin E_{\underline{r}}$ almost surely.
- Having $\mathbb{E}(M_{\nu,k}) = 1$ does not guarantee that $\mathbb{P}_\mu(M_\nu > 0) > 0$, but having

  $$\sup_{k \in \mathbb{N}} \mathbb{E}(M_{k,\nu})^2 < \infty,$$

  guarantees convergence in $L^2$ and hence in $L^1$, so $\mathbb{E}(M_\nu) = 1$, and this does give $\mathbb{P}_\mu(M_\nu > 0) > 0$.

# Billard's condition: Proof

A simple calculation gives that

$$\begin{aligned}
\mathbb{E}(M_{k,\nu})^2 &= \mathbb{E}\left(\iint \frac{\chi_{F_k(\omega)}(x)\chi_{F_k(\omega)}(y)}{\mathbb{P}(x\in F_k)\mathbb{P}(y\in F_k)}\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y)\right)\\
&= \iint \frac{\mathbb{P}(x,y\in F_k(\omega))}{\mathbb{P}(x\in F_k)\mathbb{P}(y\in F_k)}\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y)\\
&= \iint \prod_{n=1}^{k} \frac{1-\mu(B(x,r_n)-\mu(B(y,r_n)+\mu(B(x,r_n)\cap B(y,r_n))}{(1-\mu(B(x,r_n)))(1-\mu(B(y,r_n)))}\,\mathrm{d}\nu(y)\,\mathrm{d}\nu(x)\\
&\approx \iint \exp\left(\sum_{n=1}^{\infty}\mu(B(x,r_n)\cap B(y,r_n))\right)\,\mathrm{d}\nu(y)\,\mathrm{d}\nu(x) = I_{\mu,\underline{r}}(\nu),
\end{aligned}$$

Where the last equality follows using Taylor approximation $1+x\approx \exp(x+O(x^2))$, and the assumption that

$$\sup_{x\in A}\sum_{n=1}^{\infty}\mu(B(x,r_n))^2<\infty.$$

Hence if $\mathrm{Cap}_{\mu,\underline{r}}(A)>0$, that is $I_{\mu,\underline{r}}(\nu)<\infty$ for some $\nu$, $\sup_k \mathbb{E}(M_{k,\nu})^2<\infty$, so there almost surely is not a full cover. $\square$

# On the proof

First difficulty: How to use the condition $I_{\mu,\underline{r}}(\nu) = \infty$ for all Borel probability measures $\nu$ supported on $A \cap X_{\mu,\underline{r}}$ to get get information on the covering property of the set $A \cap X_{\mu,\underline{r}}$.

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### Proposition (A.-Myllyoja 2026+)

Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$ and $A\subset\mathbb{R}^d$ be analytic. If for all Borel probability measures $\nu$ supported on $A$, there exists a Borel set $A'\subset A$, such that $\nu(A')=1$, and $A'\subset E_{\underline{r}}$ $\mathbb{P}_\mu$-almost surely, then $A\subset E_{\underline{r}}$ $\mathbb{P}_\mu$-almost surely.

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The proof of this result is very simple and is based on a measurable choice argument. Here Analyticity is crucial.

- This condition is quite subtle, because earlier we saw that if $\sum_n \mu(B(x, r_n)) = \infty$ for all $x \in A$, then for any Borel probability measure $\nu$ on $A$, we have $\nu(E_{\underline{r}}) = 1$, $\mathbb{P}_\mu$-almost surely.
- Crucially, however, the proposition requires the set which is covered almost surely to be deterministic (i.e. independent of $\omega$), which is not the case trivially, since $E_{\underline{r}}(\omega)$ certainly depends on $\omega$!

# On the proof

Second difficulty: Use the condition $I_{\mu,\underline{r}}(\nu) = \infty$ for a fixed measure $\nu$ to construct a deterministic set $A'$ of full measeure which is covered almost surely. This is where the assumption that $\mu$ is supported on $\mathbb{R}$ is used! In particular, a crucial step in the argument uses the fact that removing a single point divides $\mathbb{R}$ into two disjoint sets.

# Thank you!