A variational principle for the Hausdorff dimension of non-linear carpets

Roope Anttila
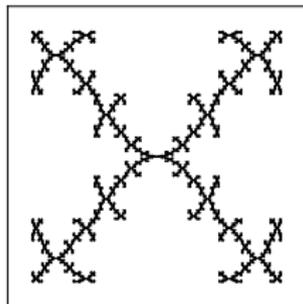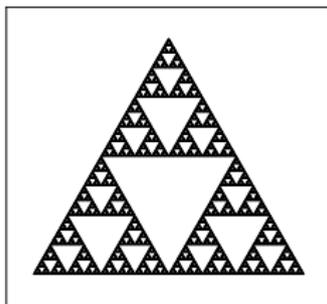
University of St Andrews

joint with

Jonathan Fraser and **Henna Koivusalo**

Statistics, hyperbolicity and related problems (SHARP)

26.03.2026

Iterated function systems
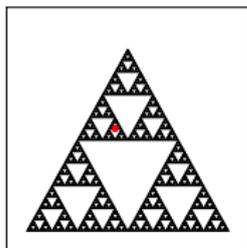

An iterated function system (IFS) is a finite collection $(S_i)_{i=1}^n$ of contracting self-maps of $\mathbb{R}^d$. The attractor of an IFS is the unique non-empty compact set which satisfies
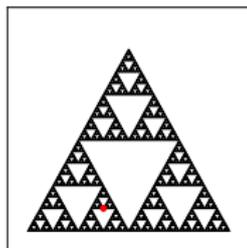
$$X = \bigcup_{i=1}^n S_i(X).$$

Objective: Quantify the geometry of attractors of IFSs.

Dynamical viewpoint

(a) A point $\pi(\mathtt{i})$

(b) The image $\pi(\sigma\mathtt{i})$

The attractor X of an IFS is an invariant set (a repeller) of a related dynamical system. There is a natural projection $\pi\colon \Sigma \to X$, where $\Sigma = \{1, \ldots, n\}^{\mathbb{N}}$ given by

$$\pi(i_1 i_2 \cdots) = \lim_{n \to \infty} S_{i_1} \circ S_{i_2} \circ \ldots \circ S_{i_n}(0).$$

We say that a measure μ on X is ergodic if there is an ergodic measure (w.r.t. σ) ν on Σ, such that $\mu = \pi_* \nu$.

Question

Does X have an ergodic measure of full Hausdorff dimension?

Variational principle for Hausdorff dimension

In some cases the answer is yes, most notably (if overlaps are not too severe) in cases where

- Each S_i is a $C^{1+\alpha}$ conformal map.
- Each S_i is an affine map (although not always!).

Sometimes finding a dimension maximising measure might be difficult (or it might not exist), but one might have a variational principle

$$\dim_H X = \sup\{\dim_H \mu \colon \mu \text{ is an ergodic measure on } X\}.$$

Goal: Establish such a variational principle for a class of IFSs which is both non-conformal and non-linear, and explain an application to a problem in Diophantine approximation on fractals.

Non-linear carpets

The class of IFSs we study are called non-linear carpets.

- Let $(f_i)_{i=1}^n$ and $(g_j)_{j=1}^m$ be self-conformal iterated function systems (IFSs) on $[0,1]$, that is, each f_i is a $C^{1+\alpha}$ contraction and each g_j is a $C^{1+\beta}$ contraction on $[0,1]$.

- We assume that both IFSs satisfy the open set condition (OSC), that is, there are open sets $U, V \subset [0,1]$, such that $f_i(U) \subset U$, $g_j(V) \subset V$, $f_{i_1}(U) \cap f_{i_2}(U) = \emptyset$ for all $i_1 \neq i_2$ and $g_{j_1}(U) \cap g_{j_2}(U) = \emptyset$ for all $j_1 \neq j_2$.

- Let $\Lambda \subset \{1, \dots, n\} \times \{1, \dots, m\}$ and consider the planar IFS $(S_{i,j} := (f_i, g_j))_{(i,j) \in \Lambda}$.

- The attractor of the IFS is called a non-linear carpet.

- For a fixed $i_0 = 1, \dots, n$, we call the collection $(S_{i_0, j})_{j: (i_0, j) \in \Lambda}$ a column, and for a fixed $j_0 = 1, \dots, n$, we call $(S_{i, j_0})_{i: (i, j_0) \in \Lambda}$ a row.
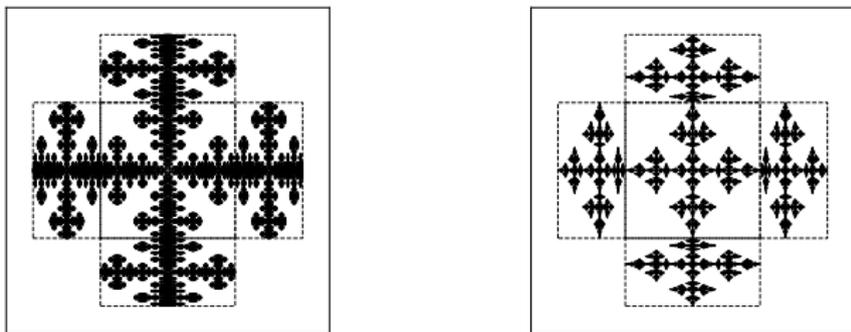
Non-linear carpets

Figure: On the left a Barański carpet and on the right a non-linear carpet.

Non-linear carpets are generalisations of the self-affine Barański carpets, which correspond to the case where $(f_i = a_i x + u_i)_{i=1}^n$ and $(g_j = b_j x + v_j)_{j=1}^m$ are self-similar IFSs which satisfy the OSC with the open set $(0,1)$.

Dimension theory of non-linear carpets

Theorem (Barański, 2007)

If X is a Barański carpet, then there exists a Bernoulli measure $\mu = \mu_{\boldsymbol{p}}$, where $\boldsymbol{p} = (p_{i,j})_{(i,j) \in \Lambda}$, on X, such that

$$\dim_H \mu = \dim_H X.$$

There is a nice formula for the Hausdorff dimension of a Bernoulli measure on X: If each $f_i = a_i x + u_i$ and $g_j = b_j x + v_j$, then if $\mu_{\boldsymbol{p}}$ is the Bernoulli measure associated with the probability vector $\boldsymbol{p} = (p_{i,j})_{(i,j) \in \Lambda}$, and we assume without loss of generality that $\lambda(\boldsymbol{p}, \boldsymbol{a}) \geqslant \lambda(\boldsymbol{p}, \boldsymbol{b})$, then

$$\dim_H \mu_{\boldsymbol{p}} = g(\boldsymbol{p}, \boldsymbol{a}, \boldsymbol{b}) = \frac{\sum_{i:(i,j) \in \Lambda} q_i(\boldsymbol{p}) \log q_i(\boldsymbol{p})}{\lambda(\boldsymbol{p}, \boldsymbol{a})} + \frac{\sum_{(i,j) \in \Lambda} p_{i,j} \log(p_{i,j}/q_i(\boldsymbol{p}))}{\lambda(\boldsymbol{p}, \boldsymbol{b})},$$

where $\lambda(\boldsymbol{p}, \boldsymbol{a}) = \sum_{(i,j) \in \Lambda} p_{i,j} \log a_i$ and $q_i(\boldsymbol{p}) = \sum_{j:(i,j) \in \Lambda} p_{i,j}$.

Note

This formula depends only on $\boldsymbol{p}$, $\boldsymbol{a}$ and $\boldsymbol{b}$, and on the measures of rectangles in each column (or row). It does not depend on translations!
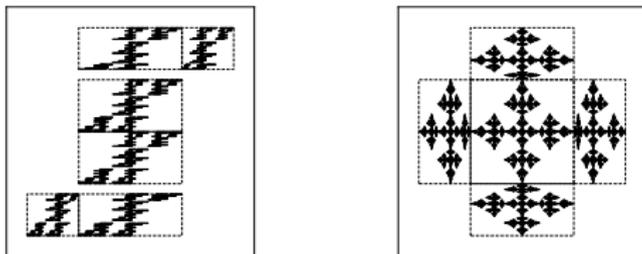
Variational principle for non-linear carpets

Figure: A dominated and a non-dominated non-linear carpet.

Theorem (A.-Fraser-Koivusalo, 2026+)

If X is a non-linear carpet, then

$$\dim_H X = \sup\{\dim_H \mu \colon \mu \text{ is ergodic}\}.$$

Assuming a domination condition, i.e. for all $(i, j) \in \Lambda$

$$\sup_{x \in [0,1]} |g_j'(x)| \leqslant \inf_{x \in [0,1]} |f_i'(x)|,$$

the analogue was proved by Gatzouras and Peres, 1997.

Idea of the proof

The idea in the proof is to approximate X by Barański carpets:

- For large $n \in \mathbb{N}$ consider the n-fold iterate of the IFS $(S_{i,j} = (f_i, g_j))_{(i,j) \in \Lambda^n}$; this has attractor X.
- For any $\epsilon > 0$, for large enough n, the bounded distortion lemma shows that

$$(\|f'_{i_1}\| \cdots \|f'_{i_k}\|)^{1+\varepsilon} \leqslant \|(f_{i_1} \circ \ldots \circ f_{i_k})'\| \leqslant \|f'_{i_1}\| \cdots \|f'_{i_k}\|$$

and similarly for $g_{j_1} \circ \ldots \circ g_{j_k}$

- Therefore X is close to being the attractor of a Barański carpet X_n with contraction ratios $a_i := \|f'_i\|$ and $b_j := \|g'_j\|$, and in particular

$$\dim_H X_n \to \dim_H X.$$

- Let ν_n be the Bernoulli measure associated with $\boldsymbol{p}_n = (p_{i,j})_{(i,j) \in \Lambda^n}$, which is the probability vector which gives the dimension maximising measure for the Barański carpet X_n
- Using similar ideas as above, one can then show that $\dim_H \mu_n \to \dim_H X$, where $\mu_n := \pi_* \nu_n$. as $n \to \infty$.
- While ν_n is only σ^n ergodic, by letting $\tilde{\mu}_n = \pi_* \tilde{\nu}_n$, where $\tilde{\nu}_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_n \circ \sigma^{-k}$, we get the claim.

Lower dimension and non-linear carpets

By using the Bernoulli measures μ_n, we can prove a rigidity result for the non-linear carpets. The lower dimension of $X \subset \mathbb{R}^d$ is

$$\dim_{\mathsf{L}} X = \sup \left\{ s > 0 \colon \exists C > 0, \text{ s.t. } \forall x \in X, 0 < r < R \leqslant 1 \right.$$

$$\left. N_r(X \cap B(x, R)) \geqslant C \left(\frac{R}{r} \right)^s \right\}.$$

- Quantifies the size of the thinnest parts of the set.
- For closed sets $\dim_{\mathsf{L}} X \leqslant \dim_{\mathsf{H}} X$ and in general (for example for most Barański carpets) the inequality is strict.

Lower dimension and non-linear carpets

> **Theorem (A.-Fraser-Koivusalo, 2026+)**
>
> *If X is a non-linear carpet, then*
>
> $$\dim_H X = \sup\{\dim_L X' : X' \subset X\}.$$

Proof combines previous methods and a trick of Ferguson–Jordan–Shmerkin.

Proof sketch: Take a deep iterate $(S_{i,j})_{(i,j) \in \Lambda^n}$ so that the attractor is close to a Barański carpet X_n and let μ_n be the dimension maximising Bernoulli measure. Take another deep iterate and pick words which have correct digit frequencies

$$\Gamma_{n,k} = \{(i_1 \cdots i_k, j_1 \cdots j_k) : \#\{i : (i_i, j_i) = (i, j)\} = \lceil k p_{i,j} \rceil \, \forall (i, j) \in \Lambda^n\}.$$

Then the attractor $X_{n,k}$ of the IFS $(S_{i,j})_{(i,j) \in \Gamma_{n,k}}$ has a very uniform structure and one can show that $\dim_L X_{n,k} \to \dim_H X_n$ as $k \to \infty$, which gives the claim since $\dim_H X_n \to \dim_H X$.

Badly approximable points

The result has an application to Diophantine approximation on fractals.

By Dirichlet's approximation theorem, for any $x \in \mathbb{R}^d$, there exist infinitely many $p \in \mathbb{R}^d$ and $q \in \mathbb{N}$, such that

$$\left\| x - \frac{p}{q} \right\| < \frac{1}{q^{1+\frac{1}{d}}}.$$

We say that a point $x \in \mathbb{R}^d$ is badly approximable if there exists $c = c(x) > 0$, such that for all $p \in \mathbb{R}^d$ and $q \in \mathbb{N}$, we have

$$\left\| x - \frac{p}{q} \right\| \geqslant \frac{c}{q^{1+\frac{1}{d}}}.$$

Question: How large is Bad_d, the set of badly approximable points in $\mathbb{R}^d$?

Theorem (Khinchine, 1926; Jarnik, 1931; Schmidt, 1969)

We have $\mathcal{L}(\mathrm{Bad}_d) = 0$ *but* $\dim_{\mathrm{H}} \mathrm{Bad}_d = d$.

Badly approximable points on fractals

- One expects Bad_d to be, in some sense, uniformly distributed in $\mathbb{R}^d$. One way to test this is to see if

$$\dim_{\mathrm{H}} X \cap \mathrm{Bad}_d = \dim_{\mathrm{H}} X,$$

  for some $X \subset \mathbb{R}^d$.

- Note that some hyperplanes contain no badly approximable numbers! Consider $X = \mathbb{R}^{d-1} \subset \mathbb{R}^d$. Then by Dirichlet's approximation theorem on $\mathbb{R}^{d-1}$, there are infinitely many $p \in \mathbb{Z}^{d-1}$ and $q \in \mathbb{N}$, such that

$$\left\| x - \frac{p}{q} \right\| < \frac{1}{q^{1+\frac{1}{d-1}}} = o\left(\frac{1}{q^{1+\frac{1}{d}}} \right),$$

  so $X \cap \mathrm{Bad}_d = \emptyset$.

- If X avoids hyperplanes, then the full dimension phenomenon has been verified for many sets, for example:

Theorem (Kleinbock-Weiss, 2005)

If X is an Ahlfors regular set which "avoids hyperplanes", then

$$\dim_{\mathrm{H}} X \cap \mathrm{Bad}_d = \dim_{\mathrm{H}} X.$$

Badly approximable points on fractals

The following result is essentially a generalisation of Kleinbock and Weiss' result to the non-Ahlfors regular setting.

> **Theorem (Fishman, 2009)**
>
> If $X \subset \mathbb{R}^d$ is closed and hyperplane diffuse, then
>
> $$\dim_H X \cap \mathsf{Bad}_d \geqslant \dim_L X.$$

Recall that if X is Ahlfors regular, then $\dim_L X = \dim_H X$, but in general $\dim_L X < \dim_H X$.

Badly approximable points on fractals

> **Theorem (A.-Fraser-Koivusalo, 2026+)**
>
> *If X is a non-linear carpet, and some row and column has at least two maps, then*
> $$\dim_H X \cap \mathrm{Bad}_2 = \dim_H X.$$

Proof: For any $\varepsilon > 0$, our earlier result gives $X' \subset X$, such that

$$\dim_L X' \geqslant \dim_H X - \varepsilon.$$

The set X' is hyperplane diffuse by the assumption. Thus

$$\dim_H X \cap \mathrm{Bad}_2 \geqslant \dim_H X' \cap \mathrm{Bad}_2 \geqslant \dim_L X' \geqslant \dim_H X - \varepsilon.$$

$\square$

Das–Fishman–Simmons–Urbański (2019) proved the analogue for Barański carpets.

Thank you!