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Quasisymmetric geometry of self-affine fractals

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LMS Harmonic analysis and PDE network
02.03.2026

Quasisymmetric mappings

A homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ is called a distortion function.

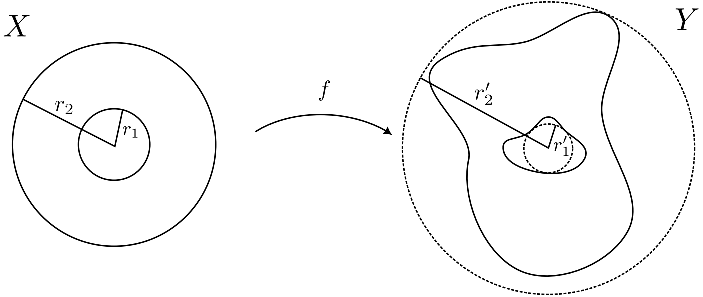
Definition

For a given distortion function η, a homeomorphism $f\colon X \to Y$ between metric spaces (X,d) and (Y,ρ) is a η-quasisymmetry if

$$\frac{\rho(f(x),f(y))}{\rho(f(x),f(z))} \leqslant \eta\left(\frac{d(x,y)}{d(x,z)}\right),$$

for all $x,y,z \in X$ with $x \neq z$. A function $f\colon X \to Y$ is called a quasisymmetry if it is an η-quasisymmetry for some distortion function η.

Quasisymmetric mappings

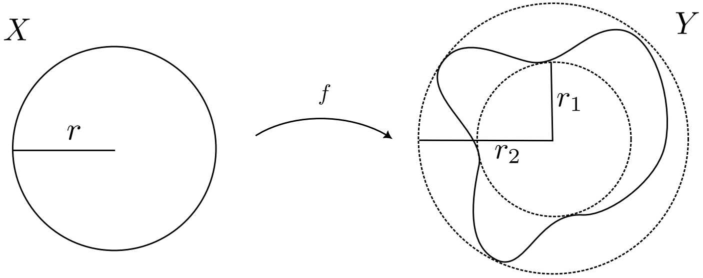


Lemma

A mapping $f : X \to Y$ *is* η*-quasisymmetric if and only if for any* $0 < r_1 \leqslant r_2$*, there exist* $0 < r'_1 \leqslant r'_2 \leqslant \eta\left(\frac{r_2}{r_1}\right) r'_1$*, and for any* $x \in X$*,*

$$B(f(x), r'_1) \subset f(B(x, r_1)) \subset f(B(x, r_2)) \subset B(f(x), r'_2).$$

Quasisymmetric mappings



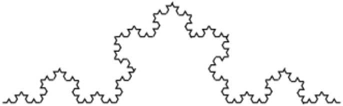
In particular, for any $r > 0$, there are $0 < r_1 \leqslant r_2 \leqslant \eta(1) r_1$, such that for any $x \in X$

$$B(f(x), r_1) \subset f(B(x, r)) \subset B(f(x), r_2).$$

Intuition

Shapes are preserved, sizes are not.

Quasisymmetric maps: Basic examples



- Every L-bi-Lipshchitz map $f\colon X \to Y$ is an η-quasisymmetry with $\eta(t) = L^2 t$:

$$\frac{\rho(f(x), f(y))}{\rho(f(x), f(z))} \leqslant \frac{Ld(x,y)}{L^{-1}d(x,z)} = L^2 \frac{d(x,y)}{d(x,z)}.$$

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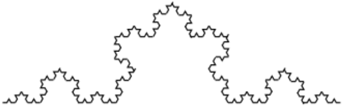

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- For any metric space (X, d) the identity map to the snowflaked space (X, d^ε) with $0 < \varepsilon < 1$ is an η-quasisymmetry with $\eta(t) = t^\varepsilon$:

$$\frac{\rho(f(x), f(y))}{\rho(f(x), f(z))} = \frac{d(x,y)^\varepsilon}{d(x,z)^\varepsilon} = \left(\frac{d(x,y)}{d(x,z)}\right)^\varepsilon$$

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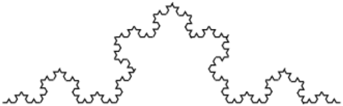

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- If $f\colon X \to Y$ and $g\colon Y \to Z$ are quasisymmetries, then $g \circ f$ is a quasisymmetry, as is f^{-1}, i.e. quasisymmetric mappings form a group.

Quasisymmetric classification

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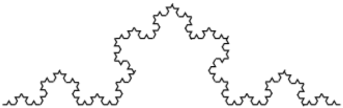

Definition

We say that metric spaces X and Y are quasisymmetrically equivalent if there exists a quasisymmetry $f\colon X \to Y$.

An important question in quasisymmetric geometry is the following:

Question

Given metric spaces X and Y, does there exist a quasisymmetry $f\colon X \to Y$.

Quasisymmetric classification

This question is motivated by a conjecture in geometric group theory.

Cannon's conjecture

If G is a hyperbolic group and ∂G is homeomorphic to S^2, then G is virtually Kleinian.

Topology of the boundary $\partial G \leftrightarrow$ Algebraic properties of G

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Cannon's conjecture 2.0

If G is a hyperbolic group and ∂G is homeomorphic to S^2, then ∂G quasisymmetrically equivalent with S^2.

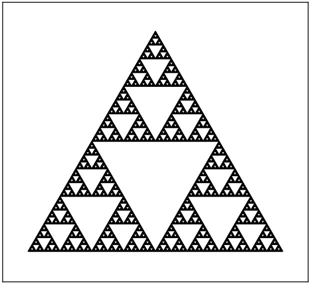
Dimension theory

- Useful quasisymmetric invariants are obtained by quantifying how quasisymmetries distort various notions of fractal dimension of a space.
- Recall that the Hausdorff dimension (denoted by $\dim_{\mathrm{H}}$) quantifies the size of a fractal by how easy the space is to cover with small balls.
- For the purpose of this talk, it suffices to think that the Hausdorff dimension of X is the unique number $s \geqslant 0$, such that

$$N_r(X) \approx r^{-s},$$

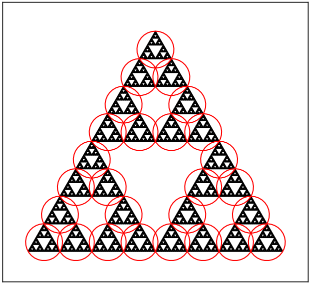
for small $r > 0$, where $N_r(X)$ is the smallest number of open balls of radius $r > 0$ needed to cover X.
- In reality this is a definition for the box dimension of a space.

Dimension theory



For example, for all $n \in \mathbb{N}$, to optimally cover the Sierpiński triangle T depicted above, by balls of radius 2^{-n}, one needs

Dimension theory



For example, for all $n \in \mathbb{N}$, to optimally cover the Sierpiński triangle T depicted above, by balls of radius 2^{-n}, one needs

$$3^n = (2^{-n})^{-\frac{\log 3}{\log 2}},$$

balls, so $\dim_H T = \frac{\log 3}{\log 2}$.

Quasisymmetric invariants

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- While bi-Lipschitz maps preserve the Hausdorff dimension of a space, the same is not true for quasisymmetries.

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- In fact, if (X, d) has positive Hausdorff dimension, then mapping (X, d) to (X, d^ε), for $0 < \varepsilon < 1$, with the identity map, we have

$$\dim_{\mathrm{H}}(X, d^\varepsilon) = \frac{1}{\varepsilon} \dim_{\mathrm{H}} X \to \infty,$$

  as $\varepsilon \to 0$. In particular, the Hausdorff dimension can be made arbitrarily large with a quasisymmetry.
- For some spaces, like $\mathbb{R}^d$, Hausdorff dimension cannot be lowered.

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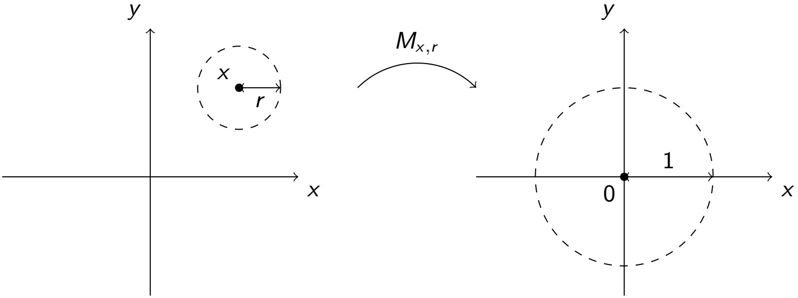
Definition

The confromal (Hausdorff) dimension of a metric space X is

$$\mathcal{C}\dim_{\mathrm{H}} X = \inf\{\dim_{\mathrm{H}} Y : Y \text{ is quasisymmetrically equivalent with } X\}.$$

- We will also be interested in another variant.

Weak tangents



For $x \in X$ and $r > 0$, define the function $M_{x,r}(y) = \frac{y-x}{r}$.

Definition

A compact set T is a weak tangent of a compact space X ($T \in \mathrm{Tan}(X)$) if there exist sequences $x_n \in X$ and $r_n > 0$, such that

$$M_{x_n,r_n}(X) \cap B(0,1) \to T$$

in the Hausdorff metric.

Assouad dimension

Definition

The Assouad dimension of a compact metric space X is

$$\dim_{\mathrm{A}} X := \max\{\dim_{\mathrm{H}} T \colon T \in \mathrm{Tan}(X)\}.$$

- The fact that the maximum is attained is result of Furstenberg.
- Quantifies the size of the "thickest parts" of the set.
- Since X is a weak tangent of itself, $\dim_{\mathrm{H}} X \leqslant \dim_{\mathrm{A}} X$, and the inequality is often strict.
- The maximal weak tangent is more regular than the original set in the following sense: If $T \in \mathrm{Tan}(X)$ satisfies $\dim_{\mathrm{H}} T = \dim_{\mathrm{A}} X$, then $\dim_{\mathrm{A}} T = \dim_{\mathrm{H}} T = \dim_{\mathrm{A}} X$.

Conformal Assouad dimension

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Assouad dimension also gives a quasisymmetric invariant.

Definition

The conformal Assouad dimension of a metric space X is

$$\mathcal{C}\dim_{\mathrm{A}} X = \inf\{\dim_{\mathrm{A}} Y \colon Y \text{ is quasisymmetrically equivalent with } X\}.$$

- We always have $\mathcal{C}\dim_{\mathrm{H}} X \leqslant \dim_{\mathrm{H}} X$, $\mathcal{C}\dim_{\mathrm{A}} X \leqslant \dim_{\mathrm{A}} X$ and $\mathcal{C}\dim_{\mathrm{H}} X \leqslant \mathcal{C}\dim_{\mathrm{A}} X$.
- All of the inequalities can be strict in general.
- Often the conformal Assouad dimension is easier to handle than the conformal Hausdorff dimension.

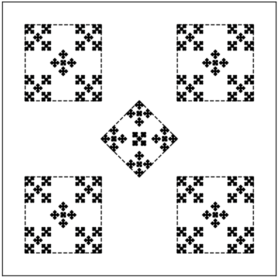
Conformal dimension in dimension theory

From a dimension theoretic point of view, for a given space X and for any notion of conformal dimension, the basic questions (in order of increasing difficulty) are the following:

- Is $\mathcal{C}\dim X < \dim X$?
- What is $\mathcal{C}\dim X$?
- Is $\mathcal{C}\dim X$ attained?

An important class of spaces are ones where the answer to the first question is no. These spaces are called minimal for conformal (Assouad, Hausdorff) dimension.

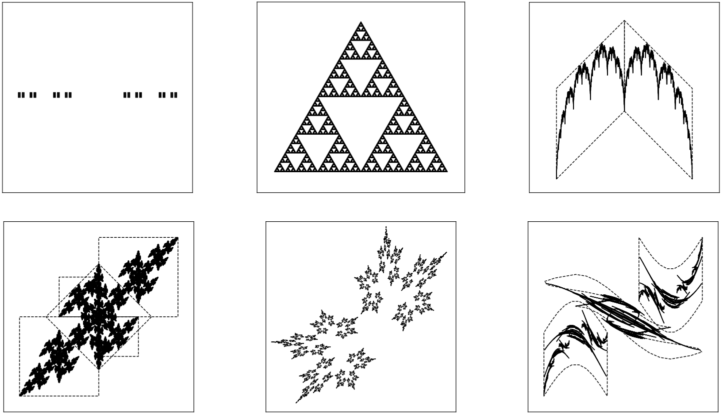
Quasisymmetric geometry of fractals



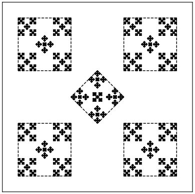
In an attempt to understand the various notions of conformal dimension better, one can look at toy models of fractals. A large class of models is given by iterated function systems (IFS). An IFS is a finite collection $(f_i)_{i=1}^n$ of contracting self-maps of $\mathbb{R}^d$. Each IFS has a unique, non-empty and compact, attractor, which is the unique set X which satisfies

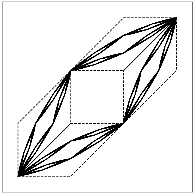
$$X = \bigcup_{i=1}^{n} f_i(X).$$

Iterated function systems: Examples



Self-similar and self-affine fractals





To simplify, we restrict ourselves to self-similar and self-affine sets:

- An IFS $(f_i)_{i=1}^n$ is self-similar if for each $i = 1, \ldots, n$,

$$f_i(x) = r_i O_i x + t_i,$$

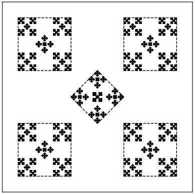
  where $0 < r_i < 1$, O_i is an orthogonal $d \times d$ matrix, and $t_i \in \mathbb{R}^d$.

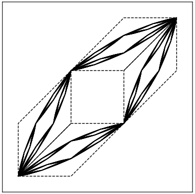
- An IFS $(f_i)_{i=1}^n$ is self-affine if for each $i = 1, \ldots, n$,

$$f_i(x) = A_i x + t_i,$$

  where A_i is an invertible $d \times d$ matrix, and $t_i \in \mathbb{R}^d$.

Quasisymmetric geometry of fractals



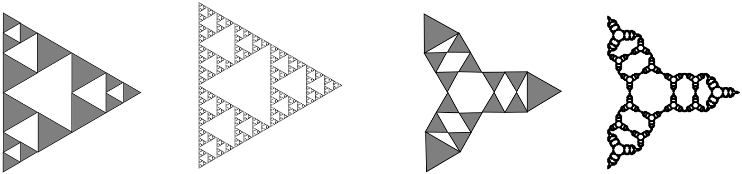


General principle

- Self-similar sets are not (usually) minimal
- Self-affine sets are (expected to be) minimal

Disclaimer: Not literally true and depends heavily on the situation!

Conformal dimension of self-similar sets



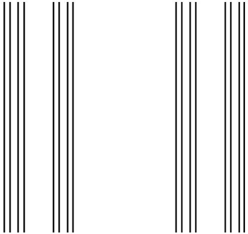
The intuition is that since self-similar sets consist of pieces exactly similar to each other, if there are not too severe overlaps, and there are gaps in between the pieces, then the pieces can be made smaller at all scales without too much distortion, so the Hausdorff dimension can usually be lowered.

Theorem (Tyson-Wu 2006)

For the Sierpiński triangle T, we have

$$\mathcal{C}\dim_{\mathrm{H}} T = \mathcal{C}\dim_{\mathrm{A}} T = 1 < \dim_{\mathrm{H}} T = \dim_{\mathrm{A}} T = \frac{\log 3}{\log 2}.$$

Minimal spaces for conformal dimension



A prototypical example of a space whose dimension cannot be lowered by a quasisymmetry is a comb.

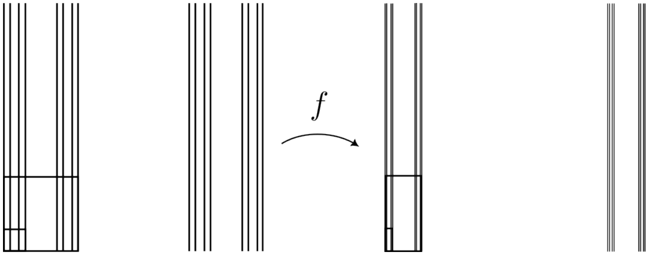
Theorem (Bishop-Tyson 2001)

For any compact set $Y \subset \mathbb{R}^d$, the space $Y \times [0, 1]$ is minimal for conformal dimensions.

General principle

Lower bounds for conformal dimension of X $\leftrightarrow$ Existence of large curve families in X

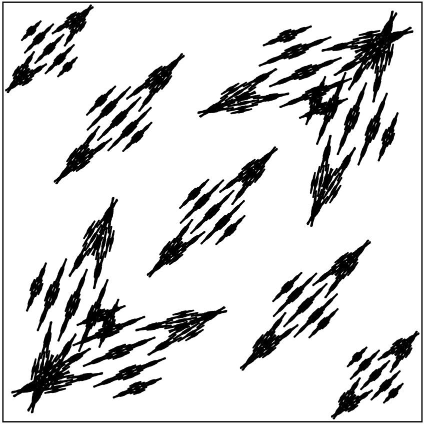
Minimal spaces for conformal dimension



Squares become increasingly eccentric!

Conformal dimension of self-affine sets

For self-affine sets, the distortion caused by the affine maps results in a local product structure.



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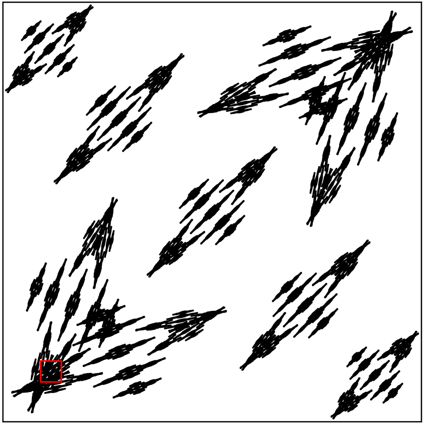

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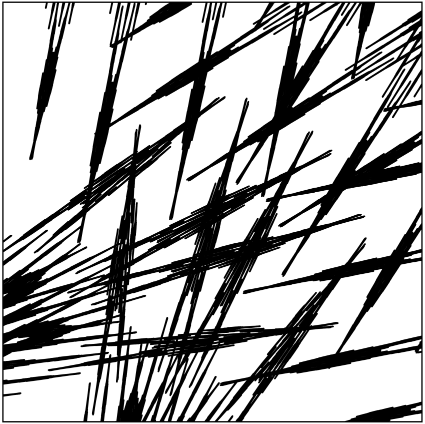

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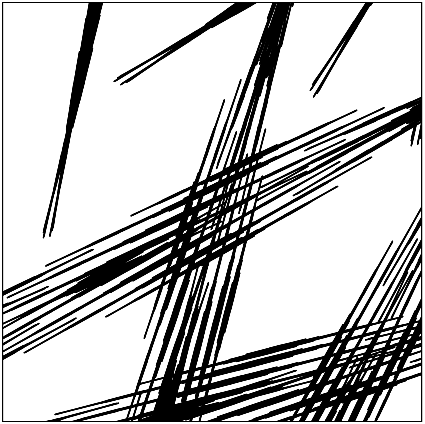

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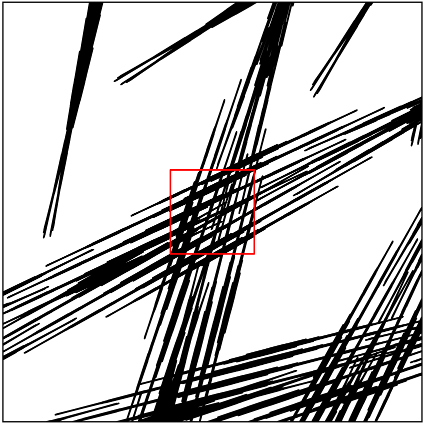

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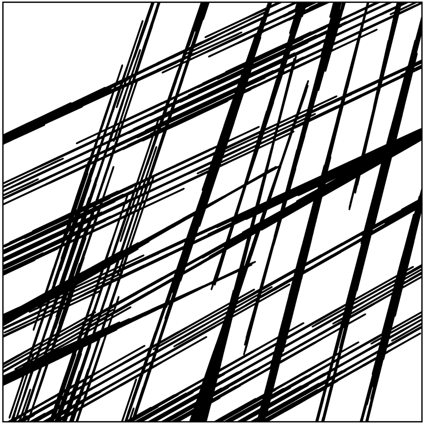

Conformal dimension of self-affine sets

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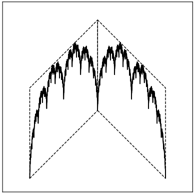 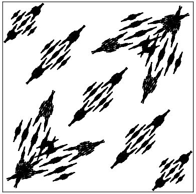

Building on earlier work by Mackay, Bárány, Käenmäki, Rossi, Yu, etc. we proved the following result.

Theorem (A.-Rutar 2026+)

If X is a weakly dominated and irreducible self-affine set, then

1. *If* $\dim_A X < 1$, *then* $\mathcal{C}\dim_A X = 0$.
2. *If* $\dim_A X \geqslant 1$, *then* $\mathcal{C}\dim_A X = \dim_A X$.

- weakly dominated: Not self-similar
- irreducible: Technical condition ensuring that the pieces don't all point in the same direction

Note that we allow for arbitrary overlaps.

On the proof

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The first item is based on the following general result.

Theorem (Tyson 2001, Kovalev 2006)

If X is a metric space with $\dim_{\mathsf{A}} X < 1$ ($\dim_{\mathsf{H}} X < 1$), then

$$\mathcal{C}\dim_{\mathsf{A}} X = 0 \ (\mathcal{C}\dim_{\mathsf{H}} X = 0).$$

The second item is proved by constructing a weak tangent which contains a comb that attains the Assouad dimension.

Thank you!