Dvoretzky covering problem on self-conformal sets

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Random covering sets

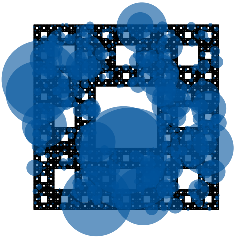

Figure: Radii: $r_n = n^{-\frac{4}{5}}$, μ: $\frac{\log 8}{\log 3}$-Hausdorff measure.

- Fix a non-increasing sequence of radii (positive real numbers) $\underline{r} = (r_n)_n$, with $r_n \downarrow 0$, and a Borel probability measure μ on $\mathbb{R}^d$.
- Place balls $B(\omega_n, r_n)$ on $\mathbb{R}^d$ where ω_n, are chosen i.i.d with respect to μ.

Random covering sets

Definition

The random covering set is the set $E_{\underline{r}} = E_{\underline{r}}(\omega)$ defined by

$$E_{\underline{r}} = \{x \in \mathbb{R}^d : x \in B(\omega_n, r_n) \text{ for infinitely many } n\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B(\omega_n, r_n).$$

Remarks:

- Formally, $E_{\underline{r}}(\omega)$ is defined for every $\omega \in (\mathbb{R}^d)^{\mathbb{N}}$, and we are interested in the properties of $E_{\underline{r}}(\omega)$ for $\mathbb{P}_\mu = \mu^{\mathbb{N}}$-typical realisations of the process.
- Assuming that r_n is non-increasing does not result in loss of generality, since we may always achieve this by reordering the sequence (using independence of the centers).
- We assume that $r_n \downarrow 0$, to ensure that $E_{\underline{r}} \subset \operatorname{spt} \mu$, $\mathbb{P}_\mu$-almost surely.

Definition

In this talk, we will focus on the case where $\underline{r} = (cn^{-\frac{1}{t}})_n$, in which case we denote the set $E_{\underline{r}}$ by $E_{c,t}$.

Random covering sets

Question

How large is $E_{\underline{r}}$ for a typical (w.r.t. $\mathbb{P}_\mu = \mu^{\mathbb{N}}$) realisation?

More specific questions include:

- Is $E_{\underline{r}} \neq \emptyset$ almost surely?
- What is $\mathcal{L}(E_{\underline{r}})$?
- What is $\dim_{\mathrm{H}} E_{\underline{r}}$?

We are interested in the following version:

Dvoretzky covering problem

When is $E_{\underline{r}} = \operatorname{spt}\mu$, $\mathbb{P}_\mu$-almost surely? More specifically, given μ, can one determine the values $c > 0$ and $t > 0$ for which $E_{c,t} = \operatorname{spt}\mu$, $\mathbb{P}_\mu$-almost surely?

Note: $\mathbb{P}_\mu(E_{\underline{r}} = \operatorname{spt}\mu) \in \{0, 1\}$ by Kolmogorov's zero-one law.

Remark

$E_{c,t} = (\neq) \operatorname{spt}\mu$ always means $E_{c,t} = (\neq) \operatorname{spt}\mu$, $\mathbb{P}_\mu$-almost surely.

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Definition

For a given μ define the critical exponent $t_{\mathrm{crit}}(\mu)$ by

$$\begin{aligned} t_{\mathrm{crit}}(\mu) &= \inf\{t > 0\colon E_{c,t} = \operatorname{spt}\mu,\ \mathbb{P}_\mu\text{-almost surely}\} \\ &= \sup\{t > 0\colon E_{c,t} \neq \operatorname{spt}\mu,\ \mathbb{P}_\mu\text{-almost surely}\} \end{aligned}$$

- If $E_{c,t} = \operatorname{spt}\mu$, then $E_{c',t'} = \operatorname{spt}\mu$ for all $c, c' > 0$ and $t' > t$, since $cn^{-\frac{1}{t}} < c'n^{-\frac{1}{t'}}$ for all large n.
- Similarly, if $E_{c,t} \neq \operatorname{spt}\mu$, then $E_{c',t'} \neq \operatorname{spt}\mu$ for all $c, c' > 0$ and $t' < t$
- $t_{\mathrm{crit}}(\mu)$ does not depend on c.
- If $0 < t_{\mathrm{crit}} < \infty$, then it is the unique number such that $E_{c,t} = \operatorname{spt}\mu$ for all $t > t_{\mathrm{crit}}$ and $E_{c,t} \neq \operatorname{spt}\mu$ for all $t < t_{\mathrm{crit}}$.

Dvoretzky covering problem

Theorem (A.-Myllyoja, 26+)

If μ is a Borel probability measure on $\mathbb{R}^d$, then

$$t_{\mathrm{crit}}(\mu) = \sup_{x \in \operatorname{spt} \mu} \underline{\dim}_{\mathrm{loc}}(\mu, x)$$

- This was proved for the Lebesgue measure on $\mathbb{T}$ by Borel (1897) and Dvoretzky (1956), for fully supported Borel probability measures on $\mathbb{T}$ by Tang (2012) and for Gibbs measures on certain topological Markov shifts by Seuret (2018).

Dvoretzky covering problem

- When $t = t_{\mathrm{crit}}(\mu)$, the covering property depends on the constant $c > 0$.
- In this case, we denote by $c_{\mathrm{crit}}(\mu)$ the critical constant by

$$\begin{aligned} c_{\mathrm{crit}}(\mu) &= \inf\{c > 0 \colon E_{c,t_{\mathrm{crit}}} = \operatorname{spt}\mu,\ \mathbb{P}_\mu\text{-almost surely}\} \\ &= \sup\{c > 0 \colon E_{c,t_{\mathrm{crit}}} \neq \operatorname{spt}\mu,\ \mathbb{P}_\mu\text{-almost surely}\}. \end{aligned}$$

- $E_{c,t_{\mathrm{crit}}} = \operatorname{spt}\mu$ for all $c > c_{\mathrm{crit}}$ and $E_{c,t_{\mathrm{crit}}} \neq \operatorname{spt}\mu$ for all $c < c_{\mathrm{crit}}$.

Theorem (Kahane 1959, Billard 1965, Erdös 1961, Fan-Karagulyan 2021)

If $\mathrm{d}\mu = f\,\mathrm{d}\mathcal{L}$ *where* $\mathcal{L}$ *is the Lebesgue measure on* $\mathbb{T}$, *then*

$$c_{\mathrm{crit}}(\mu) = \frac{1}{2\operatorname{ess\,inf}_{x\sim\mu} f(x)}.$$

Moreover, $E_{c,1} = \operatorname{spt}\mu$ *if and only if* $c \geqslant c_{\mathrm{crit}}$.

- Prior to our work, the critical constant was not known for any singular measure.

Self-conformal sets

An iterated function system (IFS) is a finite collection $\{\varphi_i\}_{i=1}^N$ of contracting self-maps of some compact subset of $\mathbb{R}$. Each IFS has a unique non-empty and compact attractor X, which satisfies

$$X = \bigcup_{i=1}^{N} \varphi_i(X).$$

We call an IFS $\{\varphi_i\}_{i=1}^N$ a self-conformal if $\varphi_i \colon [0,1] \to [0,1]$ is a contracting $C^{1+\alpha}$ map. The attractor of a self-conformal IFS is called a self-conformal set.

We will always assume that the strong separation condition (SSC) holds, i.e.

$$\varphi_i(X) \cap \varphi_j(X) = \emptyset,$$

for all $i \neq j$.

Random covering sets on self-conformal sets

We are interested in random covering sets driven by $\mu = \frac{\mathcal{H}^s|_X}{\mathcal{H}^s(X)}$, where $s = \dim_{\mathrm{H}} X$. Alternatively, μ is the *s*-conformal measure on X, that is, the unique Borel probability measure which satisfies

$$\mu(B) = \sum_{i=1}^{N} \int_{\varphi_i^{-1}(B)} |\varphi_i'(x)|^s \, \mathrm{d}\mu(x),$$

for all Borel sets $B \subset \mathbb{R}$.

Under the SSC, X is *s*-Ahlfors regular, that is, there is a constant $c > 0$, such that

$$c^{-1} r^s \leqslant \mu(B(x,r)) \leqslant c r^s.$$

Corollary

If μ is the s-conformal measure on a strongly separated self-conformal set X, then

$$t_{\mathrm{crit}}(\mu) = s.$$

Average densities of measures

It turns out that the critical constant for the Dvoretzky problem depends on the "average Ahlfors regularity constant". More precisely, we denote by

$$\overline{\mathcal{A}}(x) = \limsup_{t \to 0} \frac{1}{\log t} \int_t^1 \mu(B(x,r)) r^{-s-1} \, \mathrm{d}r,$$
$$\underline{\mathcal{A}}(x) = \liminf_{t \to 0} \frac{1}{\log t} \int_t^1 \mu(B(x,r)) r^{-s-1} \, \mathrm{d}r,$$

the upper and lower average densities of μ at x, respectively. If the limit exists it is denoted by $\mathcal{A}(x)$ and called the average density of μ at x.

Definition (Multifractal spectrum)

$$f(\alpha) := \dim_{\mathrm{H}} \{x \in X \colon \mathcal{A}(x) = \alpha\}$$

Remark: In reality, $\underline{\mathcal{A}}(x)$ is the relevant quantity, but for s-conformal measures, one has

$$\dim_{\mathrm{H}} \{x \in X \colon \underline{\mathcal{A}}(x) = \alpha\} = \dim_{\mathrm{H}} \{x \in X \colon \mathcal{A}(x) = \alpha\}.$$

Dvoretzky covering problem on self-conformal sets

Theorem (A.-Myllyoja 26+)

If μ is the s-conformal measure on a strongly separated self-conformal set X, then

$$c_{\mathrm{crit}}(\mu) = \max_{\alpha \in \mathbb{R}} \left(\frac{f(\alpha)}{s\alpha} \right)^{\frac{1}{s}}.$$

- If $c = \max_{\alpha \in \mathbb{R}} \left(\frac{f(\alpha)}{s\alpha} \right)^{\frac{1}{s}}$, we do not know if there is covering or not.
- For the natural measure on the Cantor set, we have numerical bounds.

Theorem (A.-Myllyoja, 26+)

If μ is the $\frac{\log 2}{\log 3}$-conformal measure on the Cantor set X, then

$$1.06126 \leqslant c_{\mathrm{crit}}(\mu) \leqslant 1.37546$$

Capacity and covering

For μ define the $(\mu, \underline{r})$-energy of a measure ν by

$$I_{\mu,\underline{r}}(\nu) = \iint \exp\left(\sum_{n=1}^{\infty} \mu(B(x, r_n) \cap B(y, r_n))\right) \mathrm{d}\nu(x)\,\mathrm{d}\nu(y),$$

and the $(\mu, \underline{r})$-capacity of a set $A \subset \mathbb{R}$ by

$$\mathrm{Cap}_{\mu,\underline{r}}(A) = \sup\{I_{\mu,\underline{r}}(\nu)^{-1} : \nu \text{ is a compactly supported Borel probability measure on } A\}$$

Theorem (A.-Myllyoja, 26+)

Let μ be a Borel probability measure on $\mathbb{R}$. Then $E_{\underline{r}} = \operatorname{spt}\mu$, $\mathbb{P}_\mu$-almost surely, if and only if

$$\mathrm{Cap}_{\mu,\underline{r}}(\operatorname{spt}\mu) = 0.$$

Note: For the Lebesgue measure $\mathcal{L}$ on $\mathbb{T}$, $E_{\underline{r}} = \mathbb{T}$ almost surely if and only if $I_{\mathcal{L},\underline{r}}(\mathcal{L}) = \infty$.

On the proof

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Proposition

If μ is an s-conformal measure on a strongly separated self-conformal set X and $\underline{r} = (cn^{-\frac{1}{s}})$, then $\mathrm{Cap}_{\mu,\underline{r}}(X) > 0$ if and only if there is a Borel probability measure ν on X, such that

$$\sum_{k=1}^{\infty} \nu(\varphi_{\mathtt{i}_{x,k}}(X)) \exp \sum_{n=1}^{n(k,x)} \mu(B(x, cn^{-\frac{1}{s}})) < \infty,$$

for ν-almost every x on X.

Assume for simplicity that it suffices to check measures ν supported on the level sets of the average densities which satisfy $\dim_{\mathrm{H}} \nu = f(\alpha)$. Then $\mathcal{A}(x) = \alpha$ for ν-almost every x, and

$$\nu(\varphi_{\mathtt{i}_{x,k}}(X)) \approx \|\varphi'_{\mathtt{i}_{x,k}}\|^{f(\alpha)} \text{ and } \exp \sum_{n=1}^{n(k,x)} \mu(B(x, cn^{-\frac{1}{s}})) \approx \|\varphi'_{\mathtt{i}_{x,k}}\|^{-c^S \alpha},$$

and solving $f(\alpha) - c^s\alpha = 0$ for c and optimising over ν gives the critical constant.

Capacity and covering

- For the Cantor measure μ, one has $\mathcal{A}(x) = \alpha_0 = 0.9654\ldots$, for μ-almost every $x \in X$. Moreover, one has

$$\left(\frac{f(\alpha_0)}{s\alpha_0}\right)^{\frac{1}{s}} = \left(\frac{1}{\alpha_0}\right)^{\frac{1}{s}} = 1.0573\ldots$$

- One can also show that if $c > \left(\frac{1}{\alpha_0}\right)^{\frac{1}{s}}$, then

$$I_{\mu,\underline{r}}(\mu) = \infty.$$

- Therefore, the fact that $c_{\mathrm{crit}}(\mu) \geqslant 1.06126 > \left(\frac{1}{\alpha_0}\right)^{\frac{1}{s}}$ implies that for any $\left(\frac{1}{\alpha_0}\right)^{\frac{1}{s}} < c < 1.06126$, we have $I_{\mu,\underline{r}}(\mu) = \infty$ but $\mathrm{Cap}_{\mu,\underline{r}}(X) < \infty$, so $E_{\underline{r}} \neq X$, $\mathbb{P}_\mu$-almost surely.

Upshot

The energy of the driving measure does not characterise covering in general.

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Thank you!