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Dvoretzky covering problem for general measures

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Random covering sets

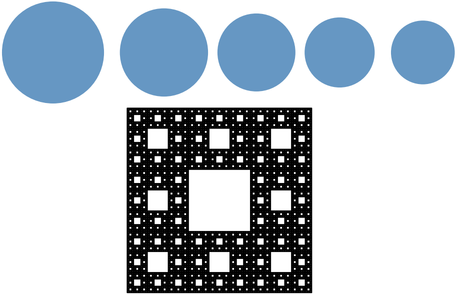

Figure: Radii: $r_n = n^{-\frac{4}{5}}$, μ: $\frac{\log 8}{\log 3}$-Hausdorff measure.

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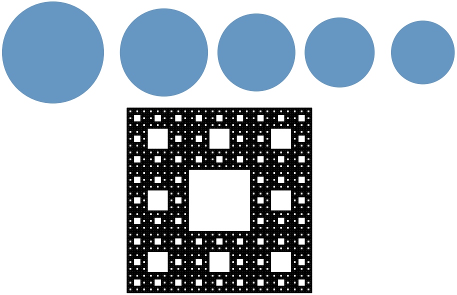

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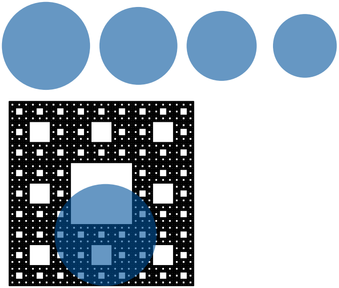

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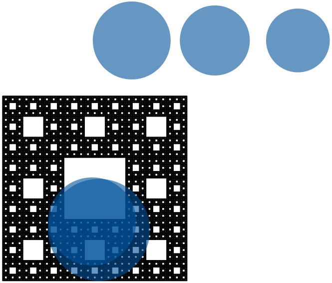

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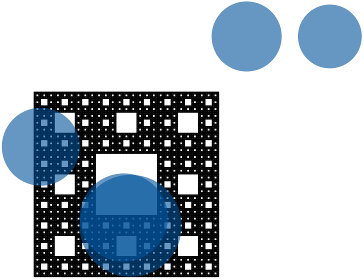

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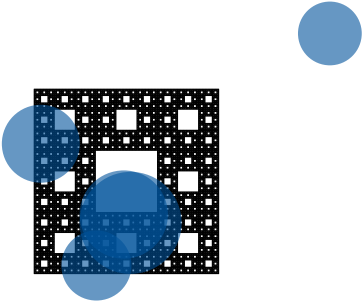

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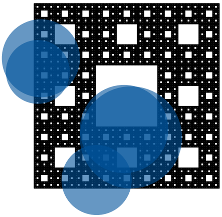

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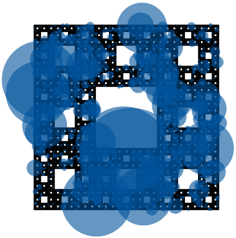

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Random covering sets

Definition

The random covering set is the set $E_{\underline{r}} = E_{\underline{r}}(\omega)$ defined by

$$E_{\underline{r}} = \{x \in \mathbb{R}^d : x \in B(\omega_n, r_n) \text{ for infinitely many } n\} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} B(\omega_n, r_n).$$

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Remarks:

- Formally, $E_{\underline{r}}(\omega)$ is defined for every $\omega \in (\mathbb{R}^d)^{\mathbb{N}}$, and we are interested in the properties of $E_{\underline{r}}(\omega)$ for $\mathbb{P}_\mu = \mu^{\mathbb{N}}$-typical realisations of the process.

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- Assuming that r_n is non-increasing does not result in loss of generality, since we may always achieve this by reordering the sequence (using independence of the centers).
- We assume that $r_n \downarrow 0$, to ensure that $E_{\underline{r}} \subset \operatorname{spt} \mu$, $\mathbb{P}_\mu$-almost surely.

Random covering sets

Connections to

- Quantitative recurrence and shrinking targets in dynamics
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Question

How large is $E_{\underline{r}}$ for a typical (w.r.t. $\mathbb{P}_\mu = \mu^{\mathbb{N}}$) realisation?

More specific questions include:

- Is $E_{\underline{r}} \neq \emptyset$ almost surely?
- What is $\mathcal{L}(E_{\underline{r}})$?
- What is $\dim_{\mathrm{H}} E_{\underline{r}}$?
- Is $E_{\underline{r}} = \operatorname{spt} \mu$ almost surely?

Dvoretzky covering problem

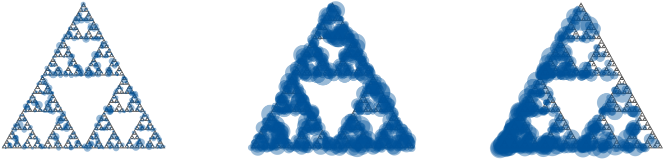
We are interested in the following generalisation of the last question:

Dvoretzky covering problem

Let μ be a Borel probability measure on $\mathbb{R}^d$ and let $A \subset \mathbb{R}^d$ be measurable. When is $A \subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely?

Note: Always $\mathbb{P}_\mu(A \subset E_{\underline{r}}) \in \{0, 1\}$ by Kolmogorov's zero-one law.

Random covering sets



(a) $\underline{r} = (n^{-\frac{4}{5}})_{n=101}^{1100}$
μ =natural measure

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(c) $\underline{r} = (n^{-\frac{3}{5}})_{n=101}^{1100}$
μ =different measure

Figure: 1000 random balls on the Sierpinski triangle

More specifically, we want to find a characterisation for when $A \subset E_{\underline{r}}$, which only depends on A, μ and $\underline{r}$.

Random covering sets: History

Classical case: $\mu = \mathcal{L}$, the Lebesgue measure on the one dimensional torus $\mathbb{T}$.

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Observation (Borel 1897)

For a fixed $x \in \mathbb{T}$, we have

1. $x \notin E_{\underline{r}}$ almost surely, if $\sum_{n=1}^{\infty} r_n < \infty$,
2. $x \in E_{\underline{r}}$ almost surely, if $\sum_{n=1}^{\infty} r_n = \infty$.

Follows from the Borel-Cantelli lemma, since

$$\mathbb{P}(x \in B(\omega_n, r_n)) = \mathbb{P}(\omega_n \in B(x, r_n)) = \mathcal{L}(B(x, r_n)) = 2r_n,$$

and since $\sum_{n=1}^{\infty} 2r_n = \infty$ if and only if $\sum_{n=1}^{\infty} r_n = \infty$.

Random covering sets: History

By Fubini's theorem, we have

$$\mathbb{E}(\mathcal{L}(E_{\underline{r}})) = \iint \chi_{E_{\underline{r}}}(x)\,\mathrm{d}x\,\mathrm{d}\mathbb{P} = \iint \chi_{E_{\underline{r}}}(x)\,\mathrm{d}\mathbb{P}\,\mathrm{d}x = \int \mathbb{P}(x \in E_{\underline{r}})\,\mathrm{d}x,$$

so Borel's observation shows that, almost surely,

$$\mathcal{L}(E_{\underline{r}}) = \begin{cases} 0, & \text{if } \sum_n r_n < \infty \\ 1, & \text{if } \sum_n r_n = \infty. \end{cases}$$

This opens up natural follow up questions, i.e. what is $\dim_{\mathrm{H}} E_{\underline{r}}$ if $\sum_n r_n < \infty$?

Dvoretzky covering problem: History

Dvoretzky 1956: Does the condition $\sum_{n=1}^{\infty} r_n = \infty$ imply that $\mathbb{T} \subset E_{\underline{r}}$ almost surely?

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Theorem (Dvoretzky 1956)

1. *If $r_n \geqslant \frac{\log n}{n}$ for all large enough n, then $\mathbb{T} \subset E_{\underline{r}}$, almost surely.*

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1. *If $r_n \geqslant \frac{\log n}{n}$ for all large enough n, then $\mathbb{T} \subset E_{\underline{r}}$, almost surely.*
2. *There exists a sequence $\underline{r} = (r_n)_n$, such that $\sum_{n=1}^{\infty} r_n = \infty$, but $\mathbb{T} \not\subset E_{\underline{r}}$, almost surely.*

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Original Dvoretzky covering problem: What is the characterising condition?

Dvoretzky covering problem: History

Borel's observation and Dvoretzky's result have the following corollary for polynomially decreasing sequences of radii $\underline{r} = (cn^{-t})_n$:

Corollary

Let $\underline{r} = (cn^{-t})_n$

1. *If* $t > 1$, *then* $\mathbb{T} \not\subset E_{\underline{r}}$ *almost surely.*
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Proof:

1. $\sum_n cn^{-t} < \infty$.
2. For all large enough $n \in \mathbb{N}$, $cn^{1-t} \geqslant \log n$, so

$$cn^{-t} \geqslant \frac{\log n}{n}.$$

$\square$

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□

Interestingly, for the critical exponent $t = 1$, the covering property depends on the constant c.

Dvoretzky covering problem: History

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Theorem

1. *(Kahane 1959) If $r_n = cn^{-1}$, for $c > \frac{1}{2}$, then $\mathbb{T} \subset E_{\underline{r}}$ almost surely.*

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Dvoretzky covering problem: History

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3. *(Erdös 1960, Mandelbrot 1972) If* $r_n = \frac{1}{2}n^{-1}$*, then* $\mathbb{T} \subset E_{\underline{r}}$ *almost surely.*

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The Dvoretzky covering problem for covering the full torus was solved by Shepp in 1972:

Theorem (Shepp 1972)

If $\underline{r} = (r_n)_n$, then $\mathbb{T}$ is covered almost surely by $E_{\underline{r}}$ if and only if

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(\sum_{k=1}^{n} 2r_k\right) = \infty.$$

This result was generalised by Kahane, who gave a potential theoretic characterisation for covering an arbitrary compact set $A \subset \mathbb{T}$.

Capacity and coverings

Let $\Phi_{\underline{r}} \colon \mathbb{T}^2 \to \mathbb{R}$ denote the kernel function

$$\Phi_{\underline{r}}(x, y) = \exp\left(\sum_{n=1}^{\infty} \max\{2r_n - |x - y|, 0\}\right) = \exp\left(\sum_{n=1}^{\infty} \mathcal{L}(B(x, r_n) \cap B(y, r_n))\right).$$

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$$I_{\underline{r}}(\nu) = \iint \Phi_{\underline{r}}(x, y) \, \mathrm{d}\nu(x) \, \mathrm{d}\nu(y),$$

and the $\underline{r}$-capacity of a set $A \subset \mathbb{T}$ is defined by

$$\mathrm{Cap}_{\underline{r}}(A) = \sup\{I_{\underline{r}}(\nu)^{-1} \colon \nu \text{ is a Borel probability measure on } A\}.$$

Here we interpret $\infty^{-1} = 0$, so $\mathrm{Cap}_{\underline{r}}(A) = 0$ if and only if $I_{\underline{r}}(\nu) = \infty$ for all measures ν on A.

Capacity and covering

Kahane proved the following:

Theorem (Kahane 1990)

A compact set $C \subset \mathbb{T}$ is covered almost surely by $E_{\underline{r}}$ if and only if

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A compact set $C \subset \mathbb{T}$ is covered almost surely by $E_{\underline{r}}$ if and only if

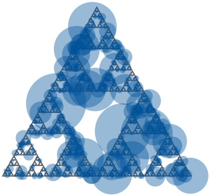
$$\mathrm{Cap}_{\underline{r}}(C) = 0.$$

- Note: One can show that $\mathrm{Cap}_{\underline{r}}(\mathbb{T}) = 0$ if and only if

$$I_{\underline{r}}(\mathcal{L}) = \iint \exp\left(\sum_{n=1}^{\infty} \max\{2r_n - |x-y|, 0\}\right) \mathrm{d}x\,\mathrm{d}y = \infty,$$

  i.e. the energy of the Lebesgue measure characterises the covering property.
- This is essentially how Shepp's proof works: He shows that $I_{\underline{r}}(\mathcal{L}) = \infty$ is equivalent with covering and that it is equivalent with the divergence of the series.

Random covering sets: General setting



For random covering sets driven by arbitrary measures μ on $\mathbb{R}^d$, basic results work essentially in the same way as for the Lebesgue measure:

Lemma

For any $x \in \mathbb{R}^d$, we have

1. $x \notin E_{\underline{r}}$ $\mathbb{P}_\mu$*-almost surely, if* $\sum_{n=1}^{\infty} \mu(B(x, r_n)) < \infty$,
2. $x \in E_{\underline{r}}$ $\mathbb{P}_\mu$*-almost surely, if* $\sum_{n=1}^{\infty} \mu(B(x, r_n)) = \infty$.

- Again follows from Borel-Cantelli, since $\mathbb{P}_\mu(x \in B(\omega_n, r_n)) = \mu(B(x, r_n))$

Random covering sets: General setting

By using Fubini as earlier,

$$\mathbb{E}(\mu(E_{\underline{r}})) = \iint \chi_{E_{\underline{r}}}(x)\,\mathrm{d}\mu(x)\,\mathrm{d}\mathbb{P} = \iint \chi_{E_{\underline{r}}}(x)\,\mathrm{d}\mathbb{P}\,\mathrm{d}\mu(x) = \int \mathbb{P}(x \in E_{\underline{r}})\,\mathrm{d}\mu(x),$$

so

$$\mu(E_{\underline{r}}) = \begin{cases} 1, & \text{if } \sum_{n=1}^{\infty} \mu(B(x, r_n)) = \infty, \text{ for all } x \in \operatorname{spt}\mu, \\ 0, & \text{if } \sum_{n=1}^{\infty} \mu(B(x, r_n)) < \infty, \text{ for all } x \in \operatorname{spt}\mu. \end{cases}$$

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Intermediate behaviour is possible.

In fact, we get something stronger: if A is a set which satisfies $\sum_n \mu(B(x, r_n)) = \infty$ for all $x \in A$, and ν is any Borel probability measure on A, then

$$\mathbb{E}(\nu(E_{\underline{r}})) = \iint \chi_{E_{\underline{r}}}(x)\,\mathrm{d}\nu(x)\,\mathrm{d}\mathbb{P} = \int \mathbb{P}(x \in E_{\underline{r}})\,\mathrm{d}\nu(x) = 1,$$

so $\nu(E_{\underline{r}}) = 1$, almost surely.

Random covering sets: General setting

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Prior results for Hausdorff dimensions of random covering sets include

- $\mu = \mathcal{L}^d$ on $\mathbb{R}^d$; arbitrary $\underline{r}$ (Jaffard 2000, Fan–Wu 2004)
- $\mu = \mathrm{vol}^d$ on Riemannian manifold; arbitrary $\underline{r}$ (Feng–Järvenpää–Järvenpää–Suomala, 2018)
- $\mu =$ Gibbs measure on topological Markov shift (self-similar measure on a homogeneous self-similar set); $\underline{r} = n^{-t}$ (Seuret, 2018)
- μ arbitrary Borel probability measure on $\mathbb{R}^d$; $\underline{r} = n^{-t}$ (Ekström–Persson, 2018)
- μ arbitrary Borel probability measure on $\mathbb{R}^d$; arbitrary $\underline{r}$ (Järvenpää–Järvenpää–Myllyoja–Stenflo, 2024; Järvenpää–Myllyoja–Seuret, 2025)
- $\vdots$

For the Dvoretzky covering problem much less was known.

Polynomially decreasing radii

For polynomially decreasing sequences, the critical exponent was known in the following cases.

Theorem (Tang 2012)

Let μ be a fully supported Borel probability measure on $\mathbb{T}$, and let $r_n = (n^{-t})_n$.

1. *If $t > (\sup_{x\in\mathbb{T}} \underline{\dim}_{\text{loc}}(\mu, x))^{-1}$, then $\mathbb{T} \not\subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely.*
2. *If $t < (\sup_{x\in\mathbb{T}} \underline{\dim}_{\text{loc}}(\mu, x))^{-1}$, then $\mathbb{T} \subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely.*

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2. *If $t < (\sup_{x\in\mathbb{T}} \underline{\dim}_{\mathrm{loc}}(\mu, x))^{-1}$, then $\mathbb{T} \subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely.*

Theorem (Seuret 2018)

Let μ be a Gibbs measure (for a Hölder potential) on an irreducible topological Markov shift Σ, and let $r_n = (n^{-t})_n$.

1. *If $t > (\max_{x\in\Sigma} \dim_{\mathrm{loc}}(\mu, x))^{-1}$, then $\Sigma \not\subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely.*
2. *If $t < (\max_{x\in\Sigma} \dim_{\mathrm{loc}}(\mu, x))^{-1}$, then $\Sigma \subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely.*

The critical constant at the critical exponent was not known for any singular measure.

Polynomially decreasing radii

Our first result is the following proposition:

Lemma (A.-Myllyoja 2026+)

Let μ be a Borel probability measure on $\mathbb{R}$ (or $\mathbb{T}$). Let $\varepsilon > 0$ and $A \subset \mathbb{R}$ be an analytic set, and assume that

$$\sum_{n=1}^{\infty} \mu(B(x, r_n))^{1+\varepsilon} = \infty,$$

for all $x \in A$. Then $A \subset E_{\underline{r}}$ almost surely.

- Proof is a quite elementary application of Borel-Cantelli.
- Recall that for $\varepsilon = 0$, the condition only implies $\nu(E_{\underline{r}}) = 1$ almost surely for all measures ν on A.

Polynomially decreasing radii

The lemma has a straightforward corollary:

Corollary (A.-Myllyoja 2026+)

Let μ *be a Borel probability measure on* $\mathbb{R}^d$, $A \subset \mathbb{R}^d$ *be analytic, and* $\underline{r} = (n^{-t})_n$.

1. *If* $t > (\sup_{x \in A} \underline{\dim}_{\mathrm{loc}}(\mu, x))^{-1}$, *then* $A \not\subset E_{\underline{r}}$, $\mathbb{P}_\mu$*-almost surely.*
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- Our earlier lemma only works in $\mathbb{R}$ but a slightly weaker variant, which is enough for the corollary, works in $\mathbb{R}^d$.
- Again the critical case $t = (\sup_{x\in A} \underline{\dim}_{\text{loc}}(\mu, x))^{-1}$ is much more subtle.

Dvoretzky covering problem: General setting

For measures other than $\mathcal{L}$, a full characterisation for the Dvoretzky covering problem was only known in the following cases:

- (Fan-Karagulyan 2021, Hirayama-Karagulyan 2025) For $\mu = f\,\mathrm{d}\mathcal{L}$, under some reqularity assumptions for f.
- (Kahane 1990) For $\mu =$ Lebesgue measure on $\mathbb{T}^d$, but for simplices (homothetic triangles in $\mathbb{R}^2$) instead of balls.

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In particular, the full characterisation was not known for any singular measure. However, a necessary condition for full covering was known in a very general setting.

Capacity and covering

For $\mathcal{L}$ define the kernel function $\Phi_{\underline{r}} \colon \mathbb{T}^2 \to \mathbb{R}$ by

$$\Phi_{\underline{r}}(x,y) = \exp\left(\sum_{n=1}^{\infty} \mathcal{L}(B(x,r_n) \cap B(y,r_n))\right),$$

and then the $\underline{r}$-energy of a measure ν by

$$I_{\underline{r}}(\nu) = \iint \Phi_{\underline{r}}(x,y)\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y),$$

and the $\underline{r}$-capacity of a set $A \subset \mathbb{T}$ by

$$\mathrm{Cap}_{\underline{r}}(A) = \sup\{I_{\underline{r}}(\nu)^{-1} \colon \nu \text{ is a Borel probability measure on } A\}.$$

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$$\mathrm{Cap}_{\mu,\underline{r}}(A) = \sup\{I_{\mu,\underline{r}}(\nu)^{-1} \colon \nu \text{ is a Borel probability measure with compact support on } A\}.$$

Billard's condition

Theorem (Billard 1965, Kahane 1985)

Let μ be a Borel probability measure on $\mathbb{R}^d$ and let $A \subset \mathbb{R}^d$ be compact, and assume that

$$\sup_{x \in A} \sum_{n=1}^{\infty} \mu(B(x, r_n))^2 < \infty.$$

If $\mathrm{Cap}_{\mu,\underline{r}}(A) > 0$, then $A \not\subset E_{\underline{r}}$, $\mathbb{P}_{\mu}$-almost surely.

Main result

For a Borel probability measure μ and a sequence of radii $\underline{r}$ we let

$$X_{\mu,\underline{r}} = \left\{ x \in \operatorname{spt}\mu \colon \sum_{n=1}^{\infty} \mu(B(x, r_n))^2 < \infty \right\}$$

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Theorem (A.-Myllyoja 2026+)

Let μ be a Borel probability measure on $\mathbb{R}$, and let $A \subset \mathbb{R}$ be analytic. Then $A \subset E_{\underline{r}}$, $\mathbb{P}_\mu$-almost surely, if and only if

$$\mathrm{Cap}_{\mu,\underline{r}}(A \cap X_{\mu,\underline{r}}) = 0.$$

- Note that for the Lebesgue measure on $\mathbb{T}$, the set $X_{\mu,\underline{r}}$ is either $\mathbb{T}$ or $\emptyset$, which is why it is not visible in Kahane's result.
- The condition depends only on A, $\underline{r}$ and μ.
- Our earlier lemma shows that $A \setminus X_{\mu,\underline{r}} = \{x \in A \colon \sum_{n=1}^{\infty} \mu(B(x, r_n))^2 = \infty\}$ is covered almost surely automatically.

Remarks

- Same result holds for measures on $\mathbb{T}$ instead of $\mathbb{R}$.
- By working in $\mathbb{R}$ instead of $\mathbb{T}$ we lose two properties which made earlier methods simpler: Compactness of the support of μ and rotational invariance.
- Even for $\mu = \mathcal{L}$ the result is new for analytic sets.

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- In general $I_{\mu,\underline{r}}(\mu) = \infty$ does not characterise covering of $\operatorname{spt}\mu$ unlike in the case of the Lebesgue measure.
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- In general $I_{\mu,\underline{r}}(\mu) = \infty$ does not characterise covering of $\operatorname{spt}\mu$ unlike in the case of the Lebesgue measure.
  - Counterexamples are given by Cantor measure and radii $cn^{-\frac{\log 3}{\log 2}}$, for some $c > 0$.
- In fact, as an application we can characterise the critical constant for the covering problem at the critical exponent for the Hausdorff measure on the Cantor set (or more generally, for the Hausdorff measure on a strongly separated self-conformal set).
- This characterisation depends on the multifractal structure of the average densities of the measure.

Billard's condition: Proof

Let $A \subset \mathbb{R}^d$ and let ν be a Borel probability measure supported on A. Denote by $F_k = F_k(\omega) = A \setminus \bigcup_{n=1}^{k} B(\omega_n, r_n)$, and consider the random variable

$$M_{\nu,k}(\omega) = \int \frac{\chi_{F_k(\omega)}(x)}{\mathbb{P}(x \in F_k)} \,\mathrm{d}\nu(x).$$

This is easily seen to be a martingale with $\mathbb{E}(M_{\nu,x}) = 1$, hence it almost surely converges (pointwise) to some random variable M_ν.

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- If $M_\nu > 0$ with positive probability, then there is a positive probability that there exists $x \in A$, such that $x \notin \bigcup_{n=1}^{\infty} B(\omega_n, r_n)$.
- But if x is not covered by any of the balls, then it is not covered infinitely often so (by the zero-one law) there is $x \in A \setminus E_{\underline{r}}$ almost surely.

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- But if x is not covered by any of the balls, then it is not covered infinitely often so (by the zero-one law) there is $x \in A \setminus E_{\underline{r}}$ almost surely.
- Having $\mathbb{E}(M_{\nu,k}) = 1$ does not guarantee that $\mathbb{P}_\mu(M_\nu > 0) > 0$, but having
$$\sup_{k\in\mathbb{N}} \mathbb{E}(M_{k,\nu})^2 < \infty,$$
guarantees convergence in L^2 and hence in L^1, so $\mathbb{E}(M_\nu) = 1$, and this does give $\mathbb{P}_\mu(M_\nu > 0) > 0$.

Billard's condition: Proof

A simple calculation gives that

$$\begin{aligned}\mathbb{E}(M_{k,\nu})^2 &= \mathbb{E}\left(\iint \frac{\chi_{F_k(\omega)}(x)\chi_{F_k(\omega)}(y)}{\mathbb{P}(x\in F_k)\mathbb{P}(y\in F_k)}\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y)\right)\\
&= \iint \frac{\mathbb{P}(x,y\in F_k(\omega))}{\mathbb{P}(x\in F_k)\mathbb{P}(y\in F_k)}\,\mathrm{d}\nu(x)\,\mathrm{d}\nu(y)\\
&= \iint \prod_{n=1}^{k} \frac{1-\mu(B(x,r_n)-\mu(B(y,r_n)+\mu(B(x,r_n)\cap B(y,r_n))}{(1-\mu(B(x,r_n)))(1-\mu(B(y,r_n)))}\,\mathrm{d}\nu(y)\,\mathrm{d}\nu(x)\\
&\approx \iint \exp\left(\sum_{n=1}^{\infty}\mu(B(x,r_n)\cap B(y,r_n))\right)\,\mathrm{d}\nu(y)\,\mathrm{d}\nu(x) = I_{\mu,\underline{r}}(\nu),\end{aligned}$$

Where the last equality follows using Taylor approximation $1+x\approx \exp(x+O(x^2))$, and the assumption that

$$\sup_{x\in A}\sum_{n=1}^{\infty}\mu(B(x,r_n))^2<\infty.$$

Hence if $\mathrm{Cap}_{\mu,\underline{r}}(A)>0$, that is $I_{\mu,\underline{r}}(\nu)<\infty$ for some ν, $\sup_k \mathbb{E}(M_{k,\nu})^2<\infty$, so there almost surely is not a full cover. □

On the proof of sufficiency

First difficulty: How to use the condition $I_{\mu,\underline{r}}(\nu) = \infty$ for all Borel probability measures ν supported on $A \cap X_{\mu,\underline{r}}$ to get get information on the covering property of the set $A \cap X_{\mu,\underline{r}}$.

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Proposition (A.-Myllyoja 2026+)

Let μ be a Borel probability measure on $\mathbb{R}^d$ and $A \subset \mathbb{R}^d$ be analytic. If for all Borel probability measures ν supported on A, there exists a Borel set $A' \subset A$, such that $\nu(A') = 1$, and $A' \subset E_{\underline{r}}$ $\mathbb{P}_\mu$-almost surely, then $A \subset E_{\underline{r}}$ $\mathbb{P}_\mu$-almost surely.

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The proof of this result is very simple and is based on a measurable choice argument. Here analyticity is crucial.

- This condition is quite subtle, because earlier we saw that if $\sum_n \mu(B(x,r_n))=\infty$ for all $x\in A$, then for any Borel probability measure ν on A, we have $\nu(E_{\underline{r}})=1$, $\mathbb{P}_\mu$-almost surely.
- Crucially, however, the proposition requires the set which is covered almost surely to be deterministic (i.e. independent of ω), which is not the case trivially, since $E_{\underline{r}}(\omega)$ certainly depends on ω!

On the proof of sufficiency

Second difficulty: Use the condition $I_{\mu,\underline{r}}(\nu) = \infty$ for a fixed measure ν to construct a deterministic set A' of full measure which is covered almost surely. This is where the assumption that μ is supported on $\mathbb{R}$ is used! In particular, a crucial step in the argument uses the fact that removing a single point divides $\mathbb{R}$ into two disjoint sets.

Thank you!