Slices of the Takagi function

Roope Anttila
joint with B. Bárány and A. Käenmäki

4.7.2023

Fractal Geometry Workshop, ICMS, Bayes Centre, Edinburgh

Takagi function

$$T_\lambda(x) = \sum_{k=0}^{\infty} \lambda^k \operatorname{dist}(2^k x, \mathbb{Z}).$$

Takagi function

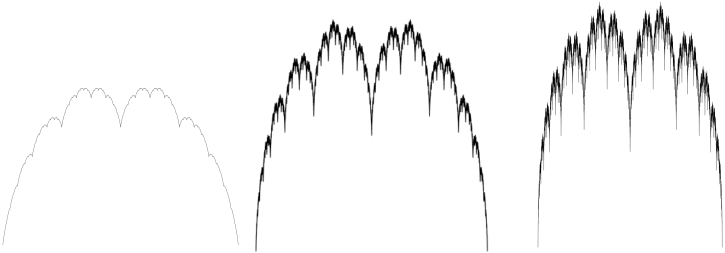


Figure: Graphs of T_λ for $\lambda = 1/2$, $\lambda = 2/3$ and $\lambda = 3/4$.

$$T_\lambda(x) = \sum_{k=0}^{\infty} \lambda^k \operatorname{dist}(2^k x, \mathbb{Z}).$$

Takagi function

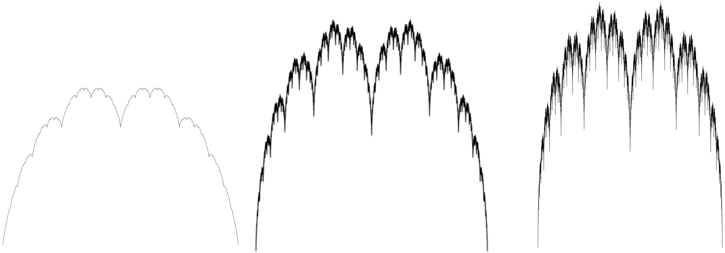


Figure: Graphs of T_λ for $\lambda = 1/2$, $\lambda = 2/3$ and $\lambda = 3/4$.

$$T_\lambda(x) = \sum_{k=0}^{\infty} \lambda^k \operatorname{dist}(2^k x, \mathbb{Z}).$$

Question

What can we say about the dimensions of the slices of T_λ?

Self-affine set

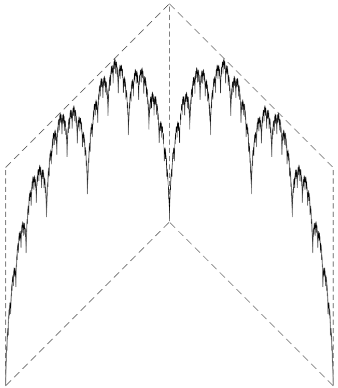


Figure: The Takagi function is an attractor of an affine IFS

Weak tangents and Assouad dimension

Let $X \subset \mathbb{R}^2$ be compact and $T_{x,r} \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a similarity taking $Q(x,r) := x + [0,r]^2$ to the unit cube $Q = [0,1]^2$ in an orientation preserving way.

Weak tangents and Assouad dimension

Let $X \subset \mathbb{R}^2$ be compact and $T_{x,r} \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a similarity taking $Q(x,r) := x + [0,r]^2$ to the unit cube $Q = [0,1]^2$ in an orientation preserving way. If there is a sequence T_{x_n,r_n}, such that $r_n \to 0$ and

$$T_{x_n,r_n}(X) \cap Q \to T$$

in the Hausdorff distance, then T is called a weak tangent of X. The collection of weak tangents of X is denoted by $\mathrm{Tan}(X)$.

Weak tangents and Assouad dimension

Let $X \subset \mathbb{R}^2$ be compact and $T_{x,r} \colon \mathbb{R}^2 \to \mathbb{R}^2$ be a similarity taking $Q(x,r) := x + [0,r]^2$ to the unit cube $Q = [0,1]^2$ in an orientation preserving way. If there is a sequence T_{x_n,r_n}, such that $r_n \to 0$ and

$$T_{x_n,r_n}(X) \cap Q \to T$$

in the Hausdorff distance, then T is called a weak tangent of X. The collection of weak tangents of X is denoted by $\mathrm{Tan}(X)$.

Theorem (Käenmäki-Ojala-Rossi, 2018)

If $X \subset \mathbb{R}^2$ is a compact set, then

$$\dim_{\mathrm{A}}(X) = \max\{\dim_{\mathrm{H}}(T) \colon T \in \mathrm{Tan}(X)\}.$$

Main result

By interpreting the Takagi function as a self-affine set, we get the following result:

Main result

By interpreting the Takagi function as a self-affine set, we get the following result:

Theorem (A.-Bárány-Käenmäki, 2023)

If T_λ is the graph of the Takagi function, with $\frac{1}{2} < \lambda < 1$, then

$$\max_{\substack{x \in T_\lambda \\ V \in \mathbb{RP}^1}} \dim_H(T_\lambda \cap (V + x)) = \dim_A(T_\lambda) - 1.$$

Main result

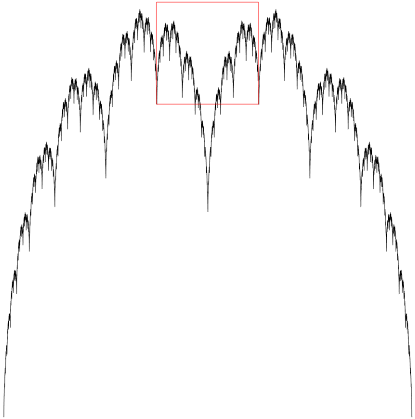
By interpreting the Takagi function as a self-affine set, we get the following result:

Theorem (A.-Bárány-Käenmäki, 2023)

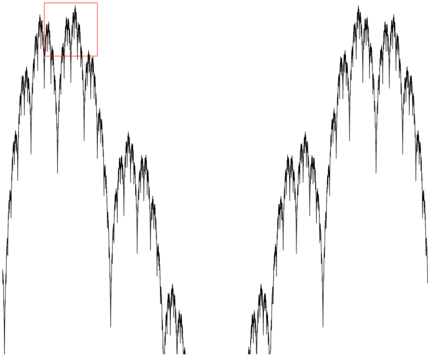
If T_λ is the graph of the Takagi function, with $\frac{1}{2} < \lambda < 1$, then

$$\dim_A(T_\lambda) = 1 + \max_{\substack{x \in T_\lambda \\ V \in \mathbb{RP}^1}} \dim_H(T_\lambda \cap (V + x)).$$

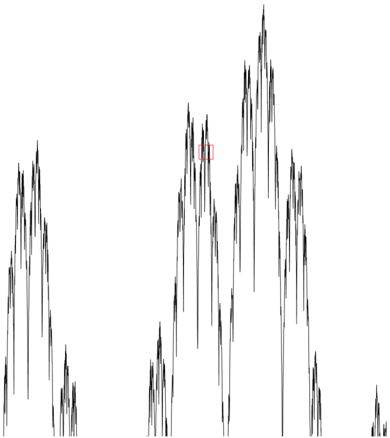
Proof



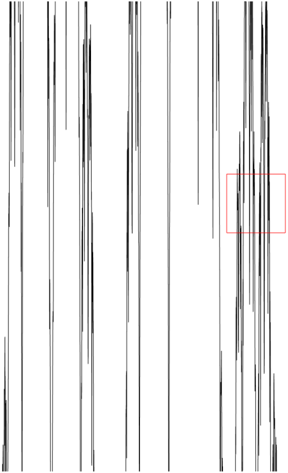
Proof



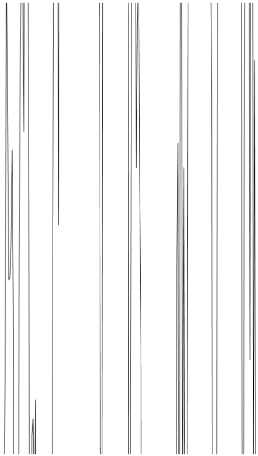
Proof



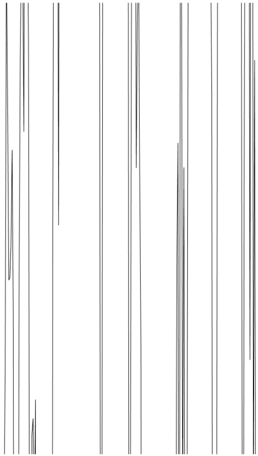
Proof



Proof

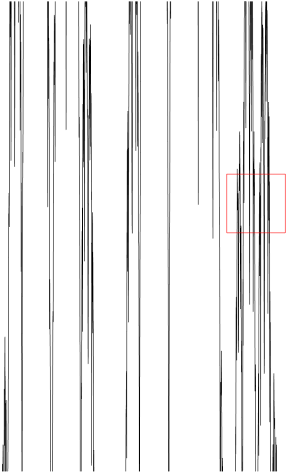


Proof

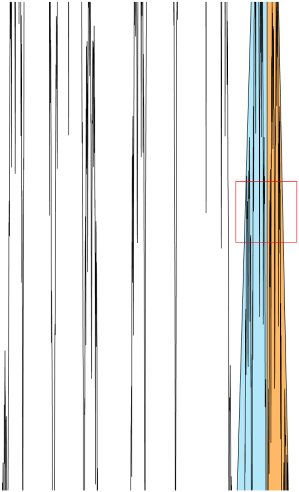


- Weak tangents are (almost) a product set.

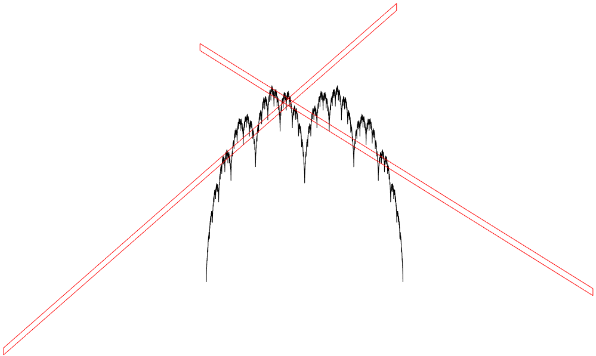
Proof



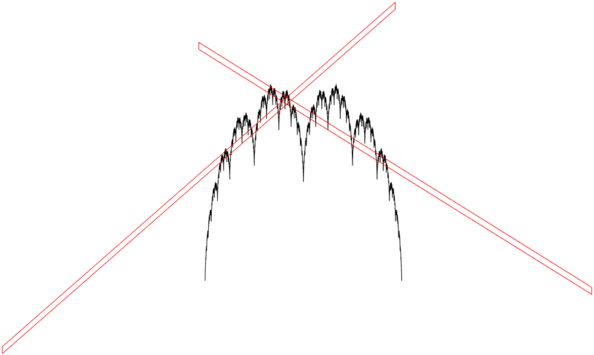
Proof



Proof

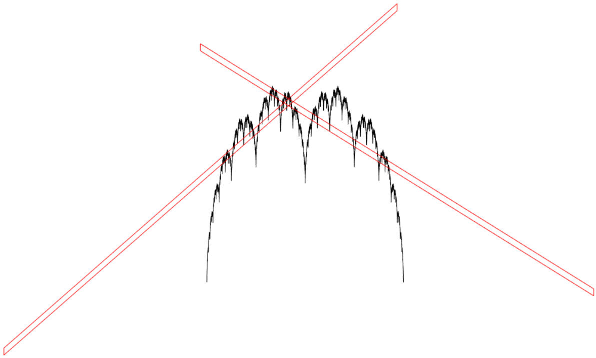


Proof



- $\implies \dim_A(T_\lambda) \leqslant 1 + \max \dim_H(T_\lambda \cap (V + x))$.

Proof



- $\implies \dim_A(T_\lambda) \leqslant 1 + \max \dim_H(T_\lambda \cap (V + x))$.
- Since the fibers are one dimensional, this turns out to be sharp. $\square$

Thank you for your attention!