



Slices of the Takagi function and (other) self-affine sets

Roope Anttila

joint with B. Bárány and A. Käenmäki

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Thermodynamic Formalism: Non-additive Aspects and Related Topics, Będlewo

Takagi function

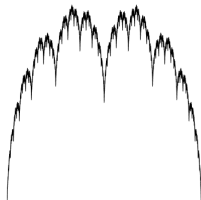


Figure: The graph of the Takagi function for $\lambda = 2/3$.

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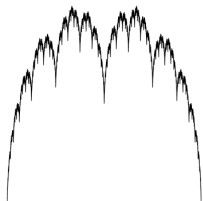


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$$T_\lambda(x) = \sum_{n=0}^{\infty} \lambda^n \text{dist}(2^n x, \mathbb{Z}).$$

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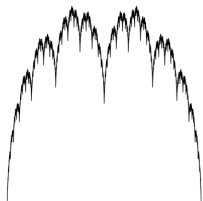


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- ▶ A well known example of a "pathological" continuous but nowhere differentiable function.

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- ▶ What can we say about **all** slices?

Main result

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Theorem (A.-Bárány-Käenmäki, 2023)

If T_λ is the graph of the Takagi function, with $\frac{1}{2} < \lambda < 1$, then

$$\max_{\substack{x \in T_\lambda \\ v \in \mathbb{R}P^1}} \dim_H(T_\lambda \cap (V + x)) = \dim_A(T_\lambda) - 1 < 1.$$

Weak tangents and Assouad dimension

Let $X \subset \mathbb{R}^2$ be compact and $T_{x,r}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a similarity taking $Q(x,r) := x + [0,r]^2$ to the unit cube $Q = [0,1]^2$ in an orientation preserving way.

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$$T_{x_n,r_n}(X) \cap Q \rightarrow T$$

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Theorem (Käenmäki-Ojala-Rossi, 2018)

If $X \subset \mathbb{R}^2$ is a compact set, then

$$\dim_A(X) = \max\{\dim_H(T) : T \in \text{Tan}(X)\}.$$

Self-affine sets

A finite collection $\{\varphi_i(x) = A_i x + t_i\}_{i=1}^M$ of invertible contractive affine maps on \mathbb{R}^2 is called a **self-affine iterated function system** (affine IFS).

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We say that X satisfies the **strong separation condition** if $\varphi_i(X) \cap \varphi_j(X) = \emptyset$, for all $i \neq j$. The associated symbolic space of infinite words is denoted by $\Sigma = \{1, \dots, M\}^{\mathbb{N}}$ and the set of finite words of length n by Σ_n . The elements of these spaces are denoted by $\mathbf{i} := (i_1, i_2, \dots)$.

First example

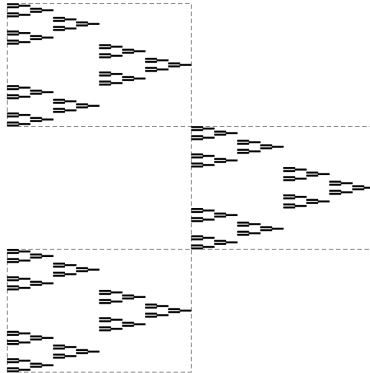


Figure: A Bedford-McMullen carpet

Second example

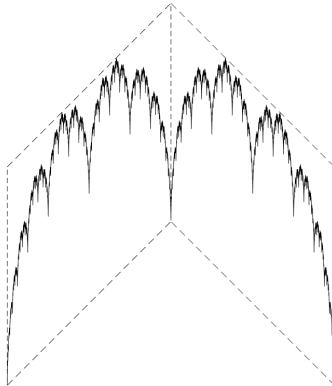
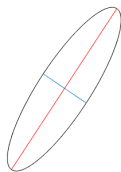
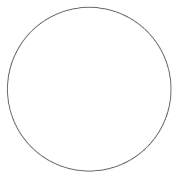
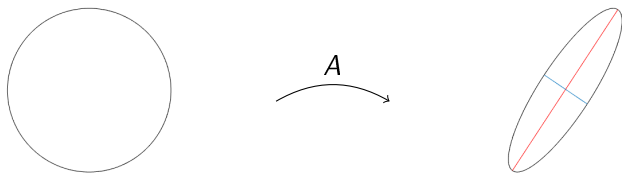


Figure: The Takagi function is an attractor of an affine IFS

Definitions

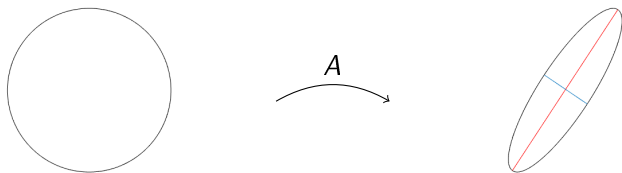


Definitions



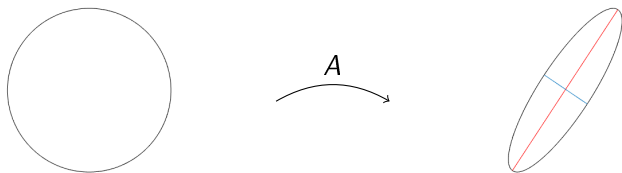
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- ▶ We assume strict inequality.
- ▶ Let $\vartheta(A)$ denote the line spanned by the longer semiaxis of $A(B(0, 1))$.

Definitions

► For $i \in \Sigma$, let

$$\bar{\vartheta}_1(i) = \lim_{n \rightarrow \infty} \vartheta(A_{i_1} \cdot \dots \cdot A_{i_n}),$$

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- ▶ Geometric interpretation: $\bar{\vartheta}_1(i)$ are the limiting directions of the construction cylinders.

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$$\frac{\alpha_2(A_{i_1} \cdot \dots \cdot A_{i_n})}{\alpha_1(A_{i_1} \cdot \dots \cdot A_{i_n})} \leq C\tau^n,$$

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If X is dominated, then the limit directions $\bar{\vartheta}_1(\mathbf{i})$ and $\bar{\vartheta}_2(\mathbf{i})$ exist for all $\mathbf{i} \in \Sigma$ and the convergence is uniform. Moreover, the sets $Y_F := \bar{\vartheta}_1(\Sigma)$ and $X_F := \bar{\vartheta}_2(\Sigma)$ are disjoint compact sets.

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- ▶ We call the sets Y_F and X_F the **forward and backward Furstenberg directions**, respectively.

Assouad dimension of self-affine sets

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Theorem (Bárány-Käenmäki-Yu, 2023)

Let X be a strongly separated, dominated self-affine set with $\dim_{\text{H}}(X) \geq 1$ and such that X_F is not a singleton. Then

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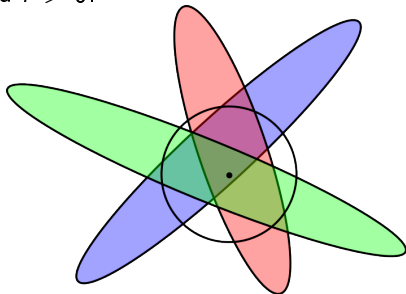
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- ▶ Strong separation condition is not satisfied.

Bounded neighbourhood condition

A self-affine set X satisfies the **bounded neighbourhood condition (BNC)** if there is a constant M , such that

$$\#\{\varphi_i \mid \alpha_2(A_i) \approx r, B(x, r) \cap \varphi_i(X) \neq \emptyset\} \leq M,$$

for all $x \in X$ and $r > 0$.



Bounded neighbourhood condition

Theorem (A.-Bárány-Käenmäki, 2023)

If X is a dominated self-affine set satisfying the BNC, such that $\dim_{\mathbb{H}}(\text{proj}_{V^{\perp}} X) = 1$ for all $V \in X_F$, then

$$\begin{aligned} \dim_{\mathbb{A}}(X) &= 1 + \max_{\substack{x \in X \\ V \in X_F}} \dim_{\mathbb{H}}(X \cap (V + x)) \\ &= 1 + \max_{\substack{x \in X \\ V \in \mathbb{R}P^1 \setminus Y_F}} \dim_{\mathbb{A}}(X \cap (V + x)). \end{aligned}$$

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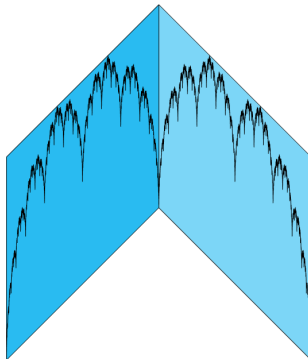
- ▶ The Takagi function is a dominated self-affine set and satisfies the BNC.
- ▶ Since T_λ is continuous, it projects to a line segment in all directions and in particular $\dim_{\mathbb{H}}(\text{proj}_{V^\perp} T_\lambda) = 1$, for all $V \in \mathbb{RP}^1$.

Proof: Upper bound

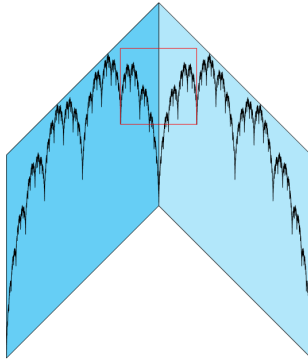
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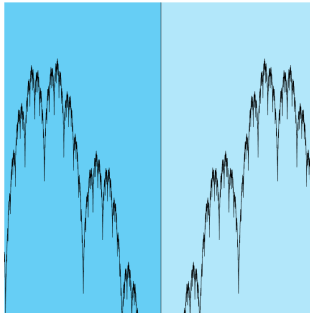
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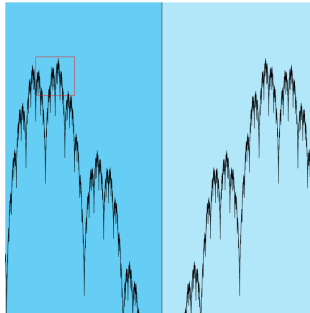
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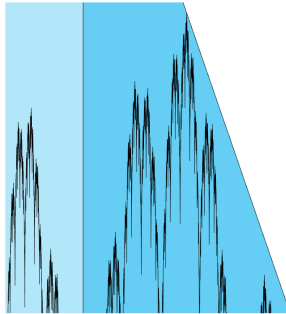
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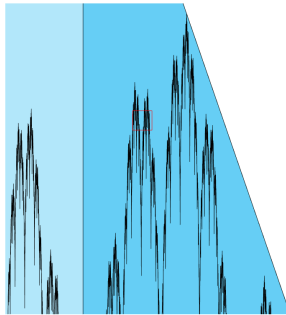
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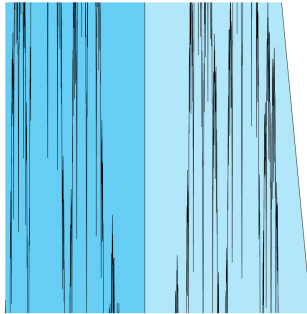
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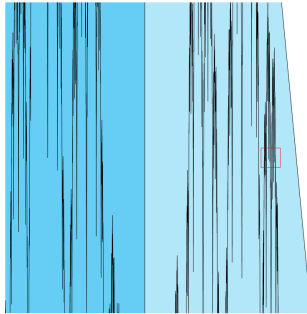
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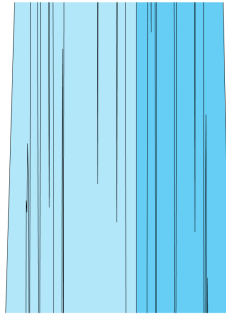
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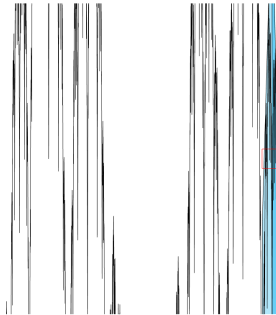
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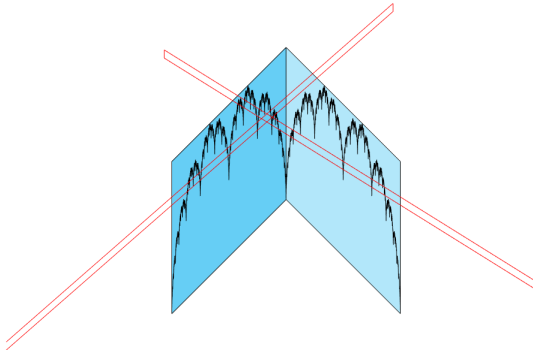
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- ▶ $\dim_{\text{H}}(T) \leq 1 + \dim_{\text{H}}(X \cap (V + x))$, where $V \in X_F$.
- ▶ $\dim_{\text{A}}(X) \leq 1 + \max_{\substack{x \in X \\ V \in X_F}} \dim_{\text{H}}(X \cap (V + x))$.

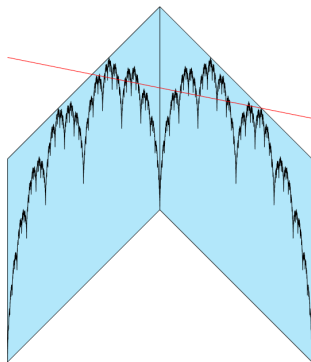


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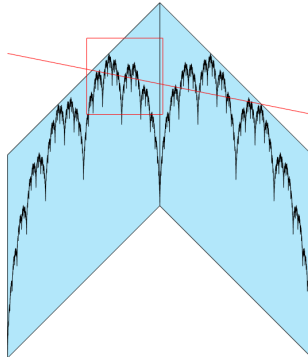
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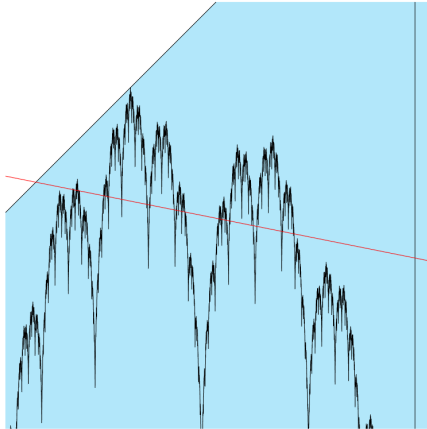
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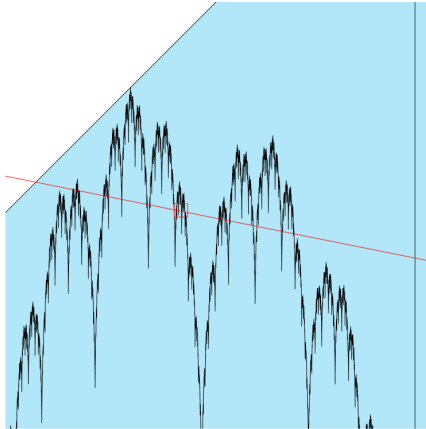
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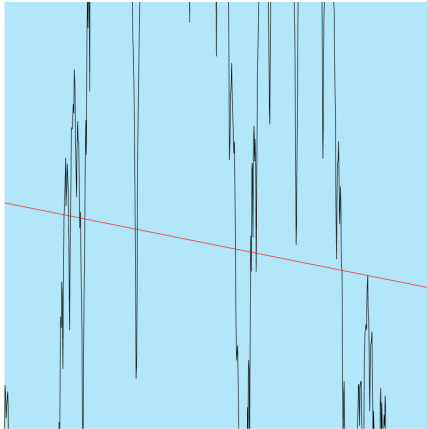
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- ▶ Take any line $V \in \mathbb{RP}^1 \setminus Y_F$ and $x \in X$.
- ▶ Let S be a weak tangent of the slice $X \cap (V + x)$ of maximal dimension.

Proof: Lower bound

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- ▶ Then $S \subset T \cap (V + y)$, where $T \in \text{Tan}(X)$.

Proof: Lower bound

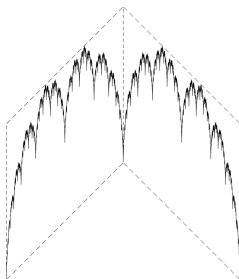
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Proof: Lower bound

- ▶ Take any line $V \in \mathbb{R}P^1 \setminus Y_F$ and $x \in X$.
- ▶ Let S be a weak tangent of the slice $X \cap (V + x)$ of maximal dimension.
- ▶ Then $S \subset T \cap (V + y)$, where $T \in \text{Tan}(X)$.
- ▶ Fiber structure $\implies "S \times \text{interval} \subset T"$
- ▶ $\dim_A(X) \geq \dim_H(T) \geq 1 + \dim_H(S) = 1 + \dim_A(X \cap (V + x)) \geq 1 + \dim_H(X \cap (V + x))$.



Back to the Takagi function



Theorem (A.-Bárány-Käenmäki, 2023)

If T_λ is the graph of the Takagi function, with $\frac{1}{2} < \lambda < 1$, then

$$\max_{\substack{x \in T_\lambda \\ V \in \mathbb{R}P^1}} \dim_H(T_\lambda \cap (V + x)) = \dim_A(T_\lambda) - 1 < 1.$$

Extending Marstrand's theorem

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 1. $\dim_{\mathbb{A}}(\mathcal{T}_\lambda) = \dim_{\mathbb{H}}(\mathcal{T}_\lambda)$, and the bound of Marstrand's slicing theorem extends to *all slices*.

Extending Marstrand's theorem

- ▶ Our theorem leaves two possibilities:
 1. $\dim_A(T_\lambda) = \dim_H(T_\lambda)$, and the bound of Marstrand's slicing theorem extends to *all slices*.
 2. $\dim_A(T_\lambda) > \dim_H(T_\lambda)$ and there is at least one slice which fails Marstrand's slicing theorem.

Extending Marstrand's theorem

Theorem (A.-Bárány-Käenmäki, 2023)

If T_λ is the graph of the Takagi function, with $\frac{1}{2} < \lambda < 1$, and μ is the projection of the uniform Bernoulli measure on the symbolic space to T_λ , then $\dim_A(T_\lambda) = \dim_H(T_\lambda)$ and in particular

$$\max_{\substack{x \in T_\lambda \\ v \in \mathbb{RP}^1}} \dim_H(T_\lambda \cap (V + x)) = \dim_H(T_\lambda) - 1,$$

if and only if

$$\overline{\dim}_{\text{loc}}(\text{proj}_{V^\perp * } \mu, \text{proj}_{V^\perp}(x)) \geq 1,$$

for all $x \in T_\lambda$ and $V \in X_F$.

About the proof

- ▶ The proof is a technical geometric argument, which establishes a connection between the local dimensions of the projected measure and the box dimensions of the slices of the set along the fibers.

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- ▶ The proof is a technical geometric argument, which establishes a connection between the local dimensions of the projected measure and the box dimensions of the slices of the set along the fibers.
- ▶ Unfortunately, we do not know any values of λ where either of the conditions hold.

Thank you for your attention!
Questions are welcome!