Slices of the Takagi function

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Paper:

Motivation

Definition

The **Takagi function** $T_\lambda\colon [0,1] \to \mathbb{R}$ is defined by

$$T_\lambda(x) = \sum_{k=0}^{\infty} \lambda^k \operatorname{dist}(2^k x, \mathbb{Z}),$$

for $\lambda \in (0,1)$.

- Different aspects of the Takagi functions, such as their (Hausdorff, packing, box) dimensions, points of differentiability, and size (cardinality, dimension) of level sets have been widely studied in the recent years.
- This project was motivated by the following question:

Question

What can be said about the size of the slices of T_λ by lines?

- Partial answers have been given:

Theorem (Marstrand's slicing theorem)

For any $\lambda \in (0,1)$, we have

$$\dim_{\mathrm{H}}(T_\lambda \cap (V+x)) \le \dim_{\mathrm{H}}(T_\lambda) - 1,$$

for Lebesque almost every $V \in \mathbb{RP}^1$ and $x \in T_\lambda$.

- Gives no information on concrete slices.

Theorem [Amo+11]

For $\lambda = \frac{1}{2}$, and for any $V \in \mathbb{RP}^1$ with integer slope,

$$\max_{x \in T_\lambda} \dim_{\mathrm{H}}(T_\lambda \cap (V+x)) = \tfrac{1}{2}.$$

- Gives information on concrete slices but the setting is very restricted.

Goal

Obtain results for the dimensions of concrete slices in a concrete setting.

Self-affine sets

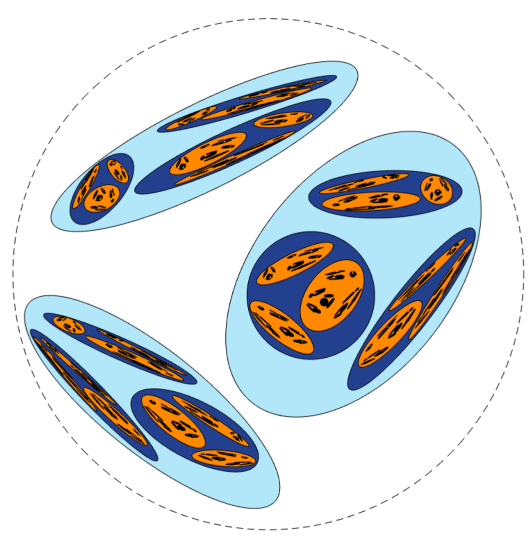


Fig. 1: The Takagi function is an example of a self-affine set

- A finite collection $\{\varphi_i(x) = A_i x + t_i\}_{i=1}^M$ of invertible contractive affine maps on $\mathbb{R}^2$ is called a **self-affine iterated function system (affine IFS)**.
- Given an affine IFS, there exists a unique, non-empty compact set X which is invariant under the IFS, that is

$$X = \bigcup_{i=1}^{M} \varphi_i(X).$$

The set X is called a **self-affine set**.

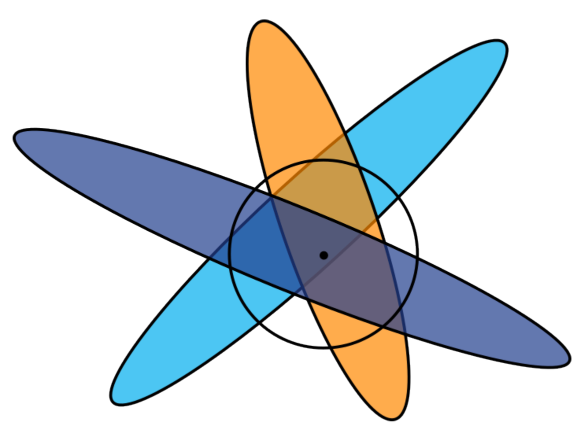


Fig. 2: Illustration of the BNC

Definition

A self-affine set X satisfies the **bounded neighbourhood condition (BNC)** if there is a constant M, such that

$$\#\{\varphi_i \mid \alpha_2(A_i) \approx r,\ B(x,r) \cap \varphi_i(X) \neq \emptyset\} \le M,$$

for all $x \in X$ and $r > 0$.

The Takagi function

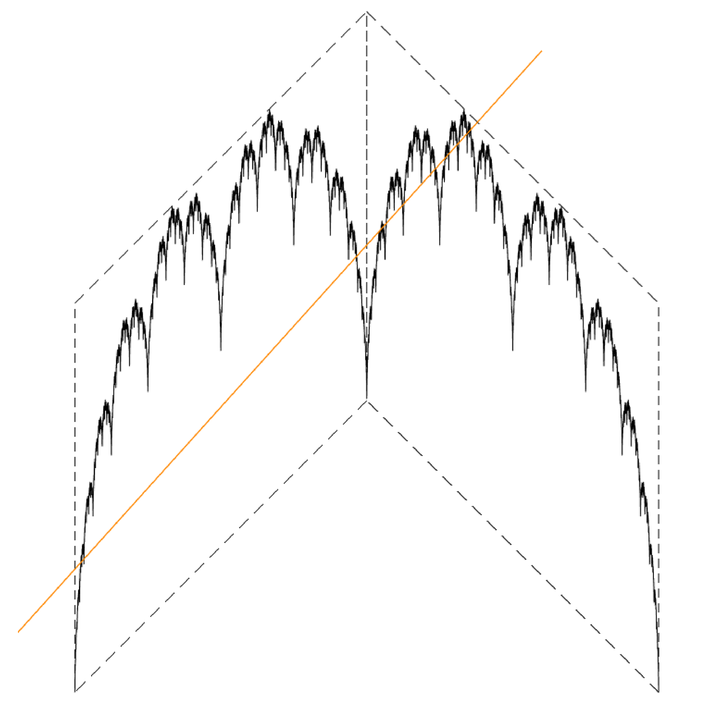
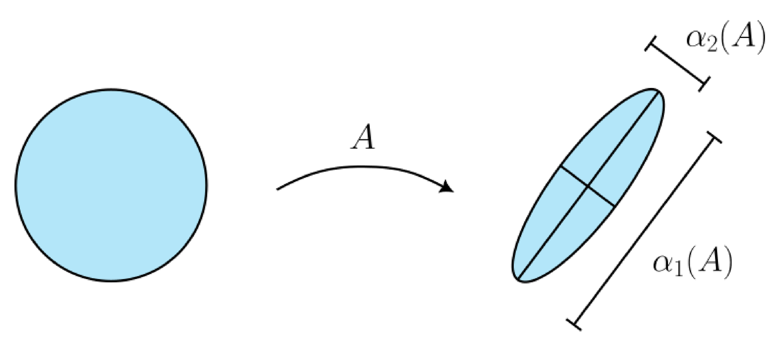


Fig. 3: A slice of the Takagi function with $\lambda = \frac{2}{3}$.

Domination



Definition

A self-affine set X is **dominated** if there exist constants $C > 0$ and $0 < \tau < 1$, such that

$$\frac{\alpha_2(A_{i_1} \cdot \ldots \cdot A_{i_n})}{\alpha_1(A_{i_1} \cdot \ldots \cdot A_{i_n})} \le C\tau^n,$$

for all $n \in \mathbb{N}$ and $(i_1, \ldots, i_n) \in \{1, \ldots, M\}^n$.

- Domination ensures that the forward and backward limit directions, denoted by Y_F and X_F, respectively, of all cylinders exist, and that $X_F \cap Y_F = \emptyset$.

Assouad dimension

Definition

The **Assouad dimension** of $X \subset \mathbb{R}^2$ is given by

$$\dim_{\mathrm{A}}(X) = \inf\Big\{ s > 0 \,|\, \exists C > 0, \text{ s.t. } \forall 0 < r < R \quad N_r(X \cap B(x,R)) \le C\left(\frac{R}{r}\right)^s \Big\},$$

where $N_r(E)$ denotes the smallest number of open balls of radius r needed to cover $E \subset \mathbb{R}^2$.

- Quantifies the size of the thickest parts of the set across all scales.
- The basic inequality is $\dim_{\mathrm{H}}(X) \le \dim_{\mathrm{A}}(X)$.

Weak tangents

Let $X \subset \mathbb{R}^2$ be compact and $T_{x,r}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a similarity taking $Q(x,r) := x + [0,r]^2$ to the unit cube $Q := [0,1]^2$ in an orientation preserving way.

Definition

If there is a sequence T_{x_n,r_n}, such that $r_n \to 0$ and

$$T_{x_n,r_n}(X) \cap Q \to T$$

in the Hausdorff distance, then T is called a **weak tangent of** X. The collection of weak tangents of X is denoted by $\operatorname{Tan}(X)$.

The weak tangets give an extremely useful characterization of the Assouad dimension.

Theorem [KOR18]

If $X \subset \mathbb{R}^2$ is compact, then

$$\dim_{\mathrm{A}}(X) = \max\{\dim_{\mathrm{H}}(T) \colon T \in \operatorname{Tan}(X)\}.$$

Main results

Theorem

If X is a dominated self-affine set satisfying the BNC, such that $\dim_{\mathrm{H}}(\operatorname{proj}_{V^\perp} X) = 1$ for all $V \in X_F$, then

$$\begin{aligned}\dim_{\mathrm{A}}(X) &= 1 + \max_{\substack{x \in X \\ V \in X_F}} \dim_{\mathrm{H}}(X \cap (V+x)) \\ &= 1 + \max_{\substack{x \in X \\ V \in \mathbb{RP}^1 \setminus Y_F}} \dim_{\mathrm{A}}(X \cap (V+x)).\end{aligned}$$

- For the Takagi functions, the following is immediate

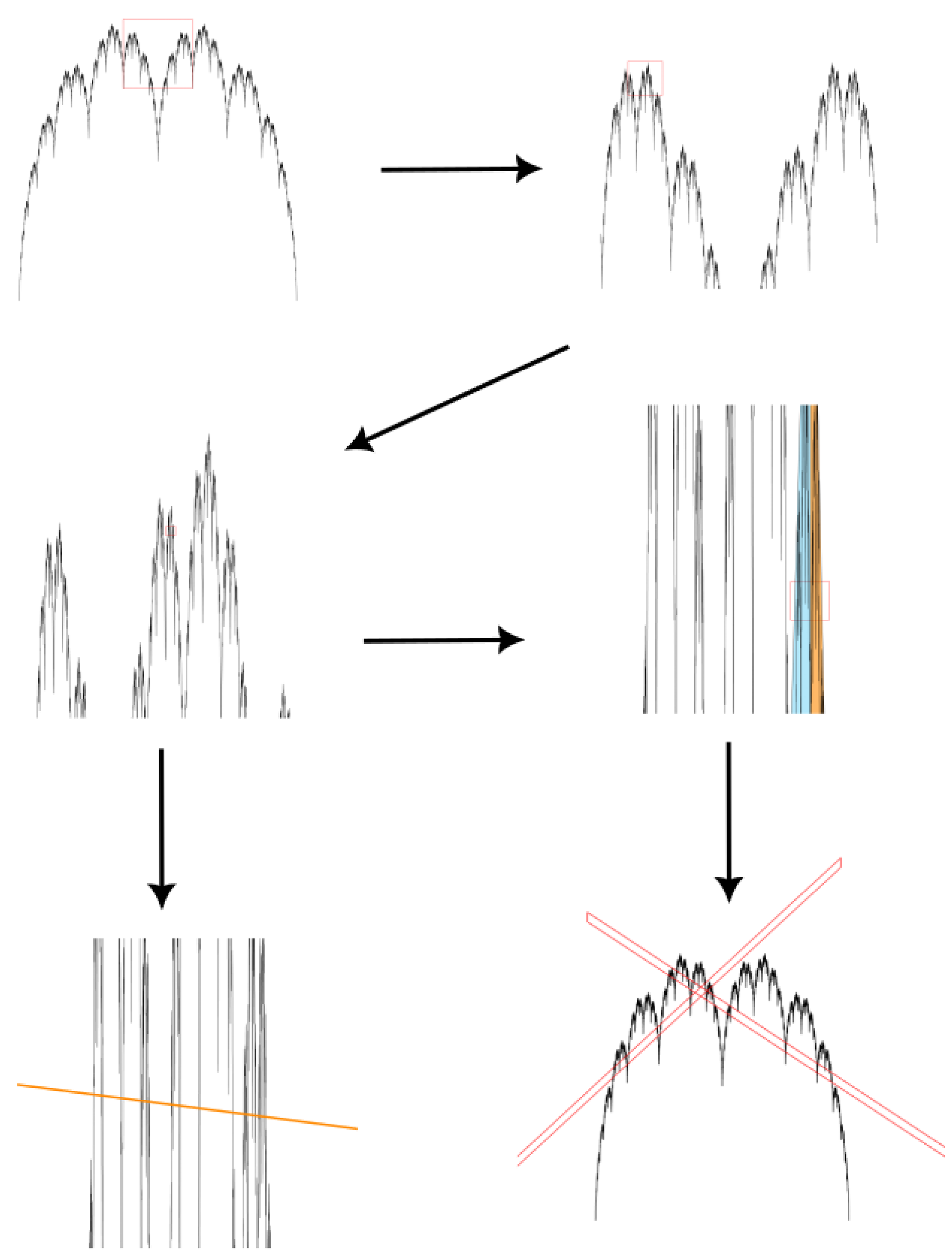
Theorem

If T_λ is the graph of the Takagi function with $\frac{1}{2} < \lambda < 1$, then

$$\max_{\substack{x \in T_\lambda \\ V \in \mathbb{RP}^1}} \dim_{\mathrm{H}}(T_\lambda \cap (V+x)) = \dim_{\mathrm{A}}(T_\lambda) - 1 < 1$$

- This achieves our goal with the caveat that the value for the Assouad dimension is not known.

Sketch of the proof



- **Upper bound:** All weak tangents have one dimensional vertical fibers and can be projected into slices of the Takagi function using inverse maps $\implies$

$$\dim_{\mathrm{A}}(T_\lambda) \le 1 + \max_{\substack{x \in T_\lambda \\ V \in \mathbb{RP}^1}} \dim_{\mathrm{H}}(T_\lambda \cap (V+x)).$$

- **Lower bound:** Weak tangents of slices of T_λ are contained in slices of weak tangents of T_λ $\implies$

$$\max_{\substack{x \in T_\lambda \\ V \in \mathbb{RP}^1}} \dim_{\mathrm{H}}(T_\lambda \cap (V+x)) \le \max_{\substack{x \in T_\lambda \\ V \in \mathbb{RP}^1}} \dim_{\mathrm{A}}(T_\lambda \cap (V+x)) \le \dim_{\mathrm{A}}(T_\lambda) - 1.$$

□

References

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[KOR18] Antti Käenmäki, Tuomo Ojala, and Eino Rossi. "Rigidity of quasisymmetric mappings on self-affine carpets". In: *Int. Math. Res. Not. IMRN* 12 (2018), pp. 3769–3799.

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