UNIVERSITY OF OULU

Pointwise Assouad dimension for measures

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Geometry of Deterministic and Random Fractals II

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Assouad dimension for measures

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Definition

The (global) Assouad dimension of μ is

$$\dim_{\mathrm{A}} \mu = \inf\left\{ s > 0 \colon \exists C > 0, \text{ s.t. } \forall 0 < r < R, \, x \in \operatorname{spt}(\mu) \; \frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant C \left(\frac{R}{r}\right)^{s} \right\}.$$

Quantifies globally the size of the least regular scales and locations of the measure.

Assouad dimension for measures

Definition

The measure μ is doubling if there exists $C > 0$, such that

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for all $x \in \operatorname{spt}(\mu)$ and $r > 0$.

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1. $\dim_H \mu \leqslant \dim_p \mu \leqslant \dim_A \mu$.
2. *μ is doubling* $\iff \dim_A \mu < \infty$.

Pointwise Assouad dimension

Quantifying the pointwise characteristics, for example by using the local dimension

$$\dim_{\mathrm{loc}}(\mu, x) := \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r},$$

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The pointwise Assouad dimension of μ at $x \in \mathrm{spt}(\mu)$ is

$$\dim_{\mathrm{A}}(\mu, x) = \inf \Big\{ s > 0 \colon \exists C(x) > 0, \text{ s.t. } \forall 0 < r < R \\ \frac{\mu(B(x,R))}{\mu(B(x,r))} \leqslant C \left(\frac{R}{r}\right)^s \Big\}.$$

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However, if the measure has nice invariant structure, then the answer to the question is positive.

Self-similar measures

- Let $\{\varphi_i(x) = r_i O_i(x) + t_i\}_{i=1}^m$, be a self-similar iterated function system (IFS) on $\mathbb{R}^d$ and let X denote the associated self-similar set.

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- We fix a probability vector $\mathrm{p} = (p_i)_{i=1}^m$ with every $p_i > 0$ and let $\mu = \mu_{\mathrm{p}}$ denote the associated self-similar measure, i.e. the unique Borel probability measure supported on X satisfying

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- We assume that X satisfies the open set condition (OSC).

Remark: Self-similar measures under the SSC are always doubling, but this is not true under OSC.

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Theorem (A. 2022)

Let μ be a self-similar measure satisfying the OSC and let ν be a fully supported ergodic measure on X. Then

$$\dim_{\mathrm{A}}(\mu, x) = \dim_{\mathrm{A}} \mu = \begin{cases} \max_{i=1,\ldots,m} \frac{\log p_i}{\log r_i}, & \textit{if } \mu \textit{ is doubling} \\ \infty, & \textit{otherwise} \end{cases},$$

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for ν-almost every $x \in X$.

Remark: The theorem holds more generally for quasi-Bernoulli measures supported on self-conformal sets and a variant holds for self-affine measures on Bedford-McMullen carpets.

About the proof

In the doubling case, the following lemma is implied by the fact that for $x = \pi(\mathtt{i})$, where $\pi\colon \Sigma \to X$ is the natural projection,

$$\mu(B(x, r_{i_1} r_{i_2} \cdots r_{i_k})) \approx p_{i_1} p_{i_2} \cdots p_{i_k},$$

Lemma

Let μ be a doubling self-similar measure with OSC. Then for any $x = \pi(\mathtt{i}) \in \mathrm{spt}(\mu)$,

$$\dim_{\mathrm{A}}(\mu, x) = \lim_{n\to\infty} \max_{k\in\mathbb{N}} \frac{\log p_{i_k} p_{i_{k+1}} \cdots p_{i_{k+n}}}{\log r_{i_k} r_{i_{k+1}} \cdots r_{i_{k+n}}}.$$

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By observing that a generic word with respect to any ergodic measure contains arbitrarily long sequences of the form $(i_{\max}, i_{\max}, \ldots, i_{\max})$, where $i_{\max}$ is the symbol maximising $\frac{\log p_i}{\log r_i}$, the claim follows.

Multifractal analysis

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Question

Given a self-similar measure satisfying the OSC, what is the multifractal spectrum of the pointwise Assouad dimension

$$f_{\mathrm{A}}(\alpha) := \dim_{\mathrm{H}}\{x \in X \colon \dim_{\mathrm{A}}(\mu, x) = \alpha\}.$$

Multifractal analysis

To simplify notation, let us define

$$E_\alpha^{\mathrm{loc}} = \{x \in X \colon \dim_{\mathrm{loc}}(\mu, x) = \alpha\},$$
$$U_\alpha^{\mathrm{loc}} = \{x \in X \colon \dim_{\mathrm{loc}}(\mu, x) \leqslant \alpha\}$$

Recall that in the classical case of the local dimensions, there exists a smooth continuous and concave function $f = f_\mu \colon [\alpha_{\min}, \alpha_{\max}] \to \mathbb{R}$, such that

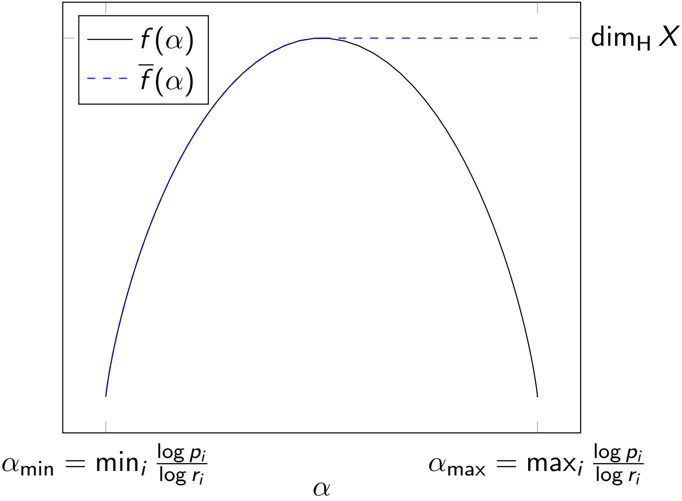
$$\dim_{\mathrm{H}} E_\alpha^{\mathrm{loc}} = f(\alpha),$$

for all $\alpha \in [\alpha_{\min}, \alpha_{\max}]$, and $E_\alpha^{\mathrm{loc}} = \emptyset$ for $\alpha \notin [\alpha_{\min}, \alpha_{\max}]$.
Moreover,

$$\dim_{\mathrm{H}} U_\alpha^{\mathrm{loc}} = \overline{f}(\alpha),$$

where $\overline{f}(\alpha) = \max_{\beta \leqslant \alpha} f(\alpha)$. Going forward, f (and thus $\overline{f}$) are fixed.

Multifractal analysis for local dimensions



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Theorem (A.-Suomala 2024)

Let μ be a non-doubling self-similar measure satisfying the OSC. Then for all $\alpha \in [\alpha_{\min}, \alpha_{\max})$, we have

$$\dim_{\mathrm{H}} E_\alpha = \dim_{\mathrm{H}} U_\alpha = \overline{f}(\alpha).$$

Moreover,

$$\dim_{\mathrm{H}} E_\infty = \dim_{\mathrm{H}} U_\infty = \dim_{\mathrm{H}} X,$$

and $E_\alpha = \emptyset$ for all $\alpha \notin [\alpha_{\min}, \alpha_{\max}] \cup \{\infty\}$.

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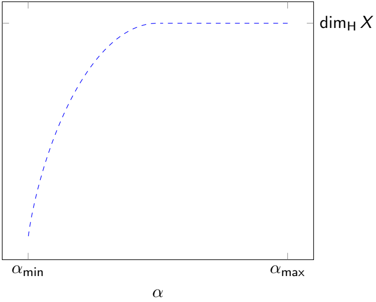

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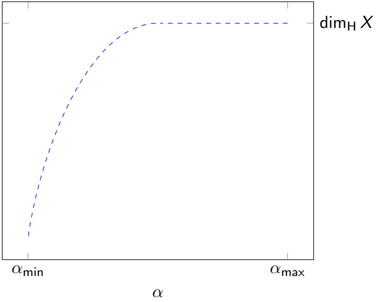

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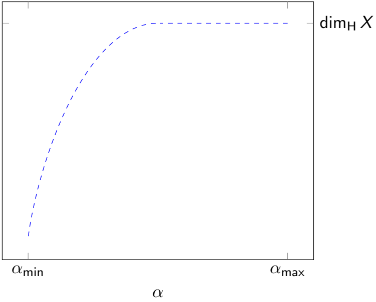

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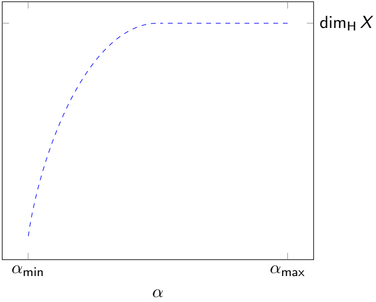

- If μ is doubling, then the multifractal spectrum fully agrees with the sub-level spectrum of the local dimension $\overline{f}$.
- If μ is not doubling, various behaviour is possible at $\alpha = \alpha_{max}$.
- If μ is doubling at x (i.e. if $\dim_A(\mu, x) < \infty$), then $\dim_A(\mu, x) \leqslant \alpha_{max}$.

About the proof

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- Because of the strong exact dimensionality, the usual measure theoretic approach is unavailable.
- Proof is combinatorial: we count the number of words with predefined frequency of symbols corresponding to a given sub-level set (method of types).
- Recalling the earlier lemma gives intuition on why the level sets have the same dimension as sub-level sets.

Lemma

Let μ be a doubling self-similar measure with OSC. Then for any $x = \pi(\mathtt{i}) \in \operatorname{spt}(\mu)$,

$$\dim_{\mathrm{A}}(\mu, x) = \lim_{n\to\infty} \max_{k\in\mathbb{N}} \frac{\log p_{i_k} p_{i_{k+1}} \cdots p_{i_{k+n}}}{\log r_{i_k} r_{i_{k+1}} \cdots r_{i_{k+n}}}.$$

Corollary

The following corollary about the size of the set of doubling points $D(\mu) := \{x \in \operatorname{spt}(\mu) \colon \mu \text{ is doubling at } x\}$ is immediate.

Corollary

Let μ be a non-doubling self-similar measure with OSC and let $s = \dim_{\mathrm{H}} X$. Then

$$\mathcal{H}^s(D(\mu)) = 0,$$

and

$$\dim_{\mathrm{H}} D(\mu) = s.$$

Thank you for your attention!

Questions are welcome!