Pointwise Assouad dimension for measures

Roope Anttila

University of Oulu, Research Unit of Mathematical Sciences

roope.anttila@oulu.fi

Introduction

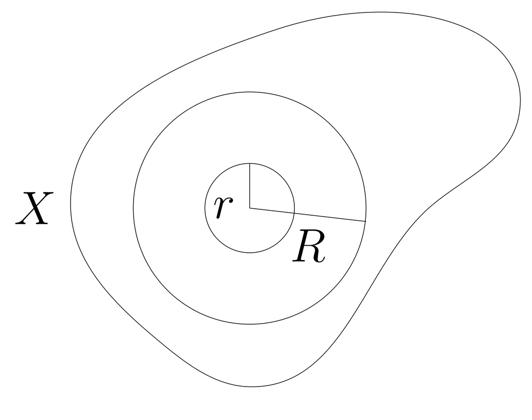


Fig. 1: A pair of concentric balls in a metric space X

We introduce a pointwise variant of the Assouad dimension for measures. Let us define our setting and recall some basics. Let μ be a finite Borel measure fully supported on a metric space X.

Definition

The **Assouad dimension** of μ is defined as

$$\dim_{\mathrm{A}} \mu = \inf \Big\{ s > 0 \mid \exists C > 0, \text{ s.t. } \forall x \in X, \; 0 < r < R, \frac{\mu(B(x,R))}{\mu(B(x,r))} \leq C \left(\frac{R}{r}\right)^s \Big\}.$$

- Quantifies the size of the most irregular parts of the measure: If a measure has a large Assouad dimension, then there are many concentric balls as in Figure 1, such that the bigger ball has much more measure than the smaller ball.
- A measure is doubling if and only if it has finite Assouad dimension.

  Recall: A measure μ is **doubling** if there exists a constant $C > 0$, such that
  $$\mu(B(x,2r)) \leq C\mu(B(x,r)),$$
  for all $x \in X$ and $r > 0$.
- The Assouad dimension is "the greatest of all dimensions" [Fra20], that is
  $$\dim_{\mathrm{A}} \mu \geq \dim \mu,$$
  where $\dim \mu$ can be replaced with Hausdorff dimension, Minkowski dimension, packing dimension, etc.

Bedford-McMullen sponges

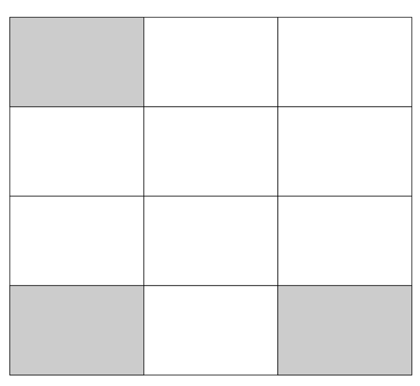


Fig. 2: An example of a Bedford-McMullen carpet satisfying the VSSC

A **Bedford-McMullen sponge** in $\mathbb{R}^d$ is constructed as follows.

1. Choose integers $n_1 < n_2 < \ldots, n_d$, and a subset $\Lambda \subset \prod_{q=1}^d \{0, \ldots, n_q - 1\}$,
2. For all $\bar{\imath} = (i_1, \ldots, i_d) \in \Lambda$, define an affine transform $\varphi_{\bar{\imath}} : [0,1]^d \to [0,1]^d$ by

$$\varphi_{\bar{\imath}}(x_1, \ldots, x_d) = \left(\frac{x_1 + i_1}{n_1}, \ldots, \frac{x_d + i_d}{n_d} \right).$$

The limit set of this IFS is called a **Bedford-McMullen sponge**.

Definition:

A Bedford-McMullen sponge satisfies the **very strong separation condition (VSSC)**, if for words $(i_1, \ldots, i_d), (j_1, \ldots, j_d) \in \Lambda$ satisfying $i_k = j_k$, for all $k = 1, \ldots, q-1$, and $i_q \neq j_q$, for some $q = 1, \ldots, d$, we have $|i_q - j_q| > 1$.(See Figure 2).

Pointwise Assouad dimension

Definition

The **pointwise Assouad dimension** of μ at $x \in X$ is given by

$$\dim_{\mathrm{A}}(\mu, x) = \inf \Big\{ s > 0 \, | \exists C_x > 0, \text{ s.t. } \forall 0 < r < R \; \frac{\mu(B(x,R))}{\mu(B(x,r))} \leq C_x \left(\frac{R}{r}\right)^s \Big\}$$

Remark: Crucially, the constant C_x in the above definition can depend on the point $x \in X$.

- Quantifies the size of the most irregular **scales** of the measure at a given point: If a measure has a large pointwise Assouad dimension, then there are many concentric balls **at a given point**, such that the bigger ball has much more measure than the smaller ball.

Main results

Proposition:

A measure is pointwise doubling at $x \in X$ if and only if it has finite pointwise Assouad dimension at x.

Recall: A measure μ is **pointwise doubling** at $x \in X$ if there exists a constant $C_x > 0$, such that
$$\mu(B(x,2r)) \leq C_x \mu(B(x,r)),$$
for all $r > 0$.

The **upper local dimension** of μ at x is defined by
$$\overline{\dim}_{\mathrm{loc}}(\mu, x) := \limsup_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r}.$$

Proposition:

We have
$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq \dim_{\mathrm{A}}(\mu, x) \leq \dim_{\mathrm{A}} \mu,$$
for all $x \in X$.

These propositions are analogous to the properties of the global Assouad dimension.

- Generally, we can have $\sup_x \dim_{\mathrm{A}}(\mu, x) < \dim_{\mathrm{A}} \mu$ (see Example).
- However in many classical examples, the pointwise Assouad dimension exhibits an exact dimensionality property.

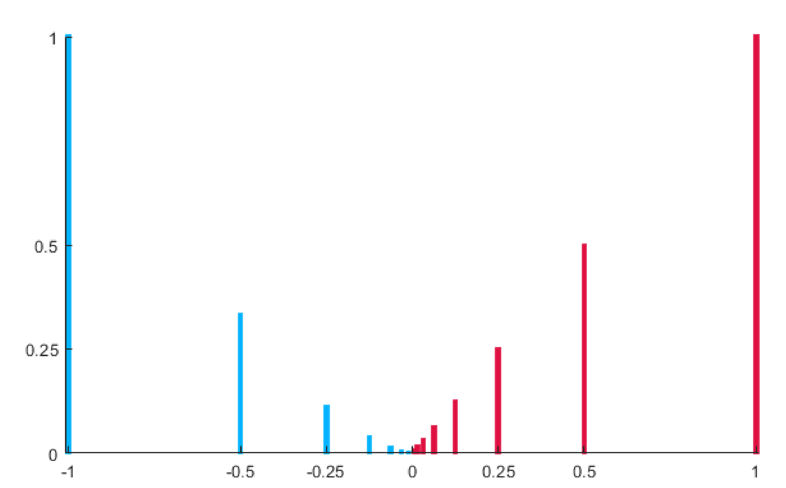
Theorem:

Let μ be one of the following:

1. A self-affine measure on a Bedford-McMullen sponge in $\mathbb{R}^d$ satisfying the VSSC,
2. A doubling self-similar measure satisfying the open set condition,
3. A quasi-Bernoulli measure on a strongly separated self-conformal set.

Then μ satisfies
$$\dim_{\mathrm{A}}(\mu, x) = \dim_{\mathrm{A}} \mu,$$
for μ-almost every $x \in \mathrm{spt}(\mu)$.

Example



The measure $\mu = \sum_{n=0}^{\infty} 3^{-n} \delta_{-2^{-n}} + 2^{-n} \delta_{2^{-n}}$ is pointwise doubling at every point in its support, but not doubling. In particular, we have $\dim_{\mathrm{A}}(\mu, x) \leq 1 < \dim_{\mathrm{A}} \mu = \infty$, for every $x \in \mathrm{spt}(\mu)$.

Self-conformal sets

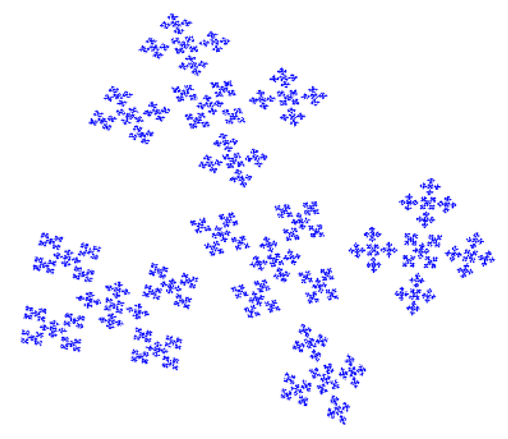


Fig. 4: An example of a self-conformal set

Let Λ be a finite index set, and let $\Sigma = \Lambda^{\mathbb{N}}$ and $\Sigma_* = \bigcup_{n=0}^{\infty} \Lambda^n$. Denote $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in \Sigma_*$.

Definition:

A measure ν on Σ is **quasi-Bernoulli**, if there is a constant $C \geq 1$, such that for all $\mathbf{i}, \mathbf{j} \in \Sigma_*$, we have
$$C^{-1}\nu([\mathbf{i}])\nu([\mathbf{j}]) \leq \nu([\mathbf{ij}]) \leq C\nu([\mathbf{i}])\nu([\mathbf{j}]).$$

An Iterated Function System (IFS) $\{\varphi_i\}_{i=1}^N$ is **conformal** if

1. There is an open, bounded and connected $\Omega \subset \mathbb{R}^d$, and a compact $X \subset \Omega$ with non-empty interior, such that
   $$\varphi_i(X) \subset X,$$
   for all $i \in \Lambda$.
2. For each $i \in \Lambda$, the map φ_i is a contracting $C^{1+\varepsilon}$-diffeomorphism, and $\varphi_i|_\Omega$ is a conformal map. That is, for every $y \in \Omega$, we have
   $$|\varphi_i'(x)y| = |\varphi_i'(x)||y|,$$
   where $|\varphi_i'(x)|$ denotes the operator norm of the linear map $\varphi_i'(x)$.

The limit set of such an IFS is called a **self-conformal set**.

If $\mu := \pi_* \nu$ is the projection of a quasi-Bernoulli measure ν on Σ under the natural projection $\pi \colon \Sigma \to F$ given by
$$\{\pi(\mathbf{i})\} = \bigcap_{n=1}^{\infty} \varphi_{\mathbf{i}|_n}(F),$$
then μ is called quasi-Bernoulli.

Future work

Next step could be to analyse the μ-null set left out in the previous Theorem by developing theory for **multifractal analysis** of the pointwise Assouad dimension. That is, one could study the size of the α-level sets
$$E_\alpha := \{x \in \mathrm{supp}(\mu) \colon \dim_{\mathrm{A}}(\mu, x) = \alpha\}.$$
More specifically, it would be interesting to calculate the **multifractal Assouad spectrum of** μ given by
$$f(\alpha) = \dim_{\mathrm{H}} E_\alpha,$$
for example in the context of self-similar or self-affine measures.

References

[Ant22] R. Anttila. "Pointwise Assouad dimension for measures". Preprint, available at `https://arxiv.org/abs/2203.15301`. 2022.

[Fra20] J.M. Fraser. *Assouad Dimension and Fractal Geometry.* Cambridge Tracts in Mathematics. Cambridge University Press, 2020.

LaTeX TikZposter