UNIVERSITY OF OULU

Assouad Dimension of Invariant Measures for Place Dependent Probabilities

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Assouad dimension of measures

Recall the definition of the Assouad dimension of a measure.

Definition

Let μ be a finite Borel probability measure fully supported on a metric space X. The *Assouad dimension of* μ is defined as

$$\dim_{\mathrm{A}} \mu = \inf \left\{ s > 0 \colon \exists C > 0, \text{ s.t. } \forall x \in X, 0 < r < R \quad \frac{\mu(B(x,R))}{\mu(B(x,r))} \leq C \left(\frac{R}{r} \right)^s \right\}$$

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(C2) For each $i \in \{1, \dots, N\}$, the map φ_i is a contractive $C^{1+\varepsilon}$-diffeomorphism, and $\varphi_i \colon \Omega \to \Omega$ is conformal, that is, $\varphi_i'(x)$ is a similarity for all $x \in \Omega$.

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The limit set F of this IFS is called a **self-conformal set**.

Example

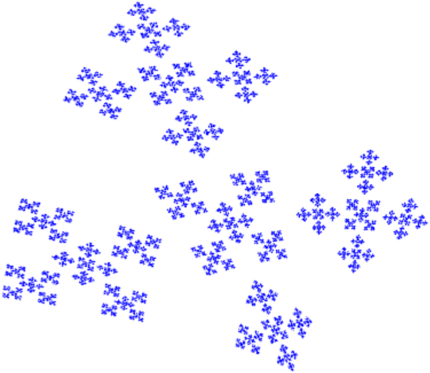

Figure: An example of a self-conformal set

Place dependent probabilities

We choose for each $i \in \{1, \dots, N\}$ a Hölder continuous function $p_i \colon X \to (0,1)$, which satisfy $\sum_{i=1}^{N} p_i(x) \equiv 1$ and consider the probability measures μ satisfying the equation

$$\int f(x) d\mu(x) = \sum_{i=1}^{N} \int p_i(x) f \circ \varphi_i(x) d\mu(x),$$

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for $f \in C(X)$ where $C(X)$ are the continuous real valued functions on X.Measures that satisfy this equation are called **invariant measures for place dependent probabilities**. Under our assumptions, this measure exists and is unique and we denote it by μ.

Notation

Let $\Sigma = \{1, \ldots, N\}^{\mathbb{N}}$ and denote $\mathtt{i} = (i_1, i_2, \ldots) \in \Sigma$. For $\mathtt{i} \in \Sigma$, let $\mathtt{i}|_n = (i_1, \ldots, i_n)$.

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$$\{\pi(\mathtt{i})\} = \bigcap_{n=1}^{\infty} \varphi_{\mathtt{i}|_n}(F).$$

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For $\mathtt{i} \in \Sigma$ and $n \in \mathbb{N}$ we let

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Denote by $P(\Sigma) \subset \Sigma$ the set of periodic points of Σ. For $\mathtt{i} \in P(\Sigma)$ with period of length n, we let

$$\overline{p}_{\mathtt{i}} = p_{\mathtt{i}|_n}(\sigma^n \mathtt{i}), \text{ and } \quad |\varphi'_{\mathtt{i}}| = |\varphi'_{\mathtt{i}|_n}(\pi(\mathtt{i}))|.$$

Results

Theorem (A. 2022)

Let μ be an invariant measure for place dependent probabilities fully supported on a strongly separated self-conformal set F. Then

$$\dim_A \mu = \sup_{\mathtt{i} \in P(\Sigma)} \frac{\log \overline{p}_{\mathtt{i}}}{\log |\varphi'_{\mathtt{i}}|}.$$

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Corollary

Let μ be a self-similar measure satisfying the SSC. Then

$$\dim_A \mu = \max_{i=1,\ldots,N} \frac{\log p_i}{\log r_i}.$$