Assouad dimension of self-affine sets

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joint with B. Bárány and A. Käenmäki

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Fractal geometry

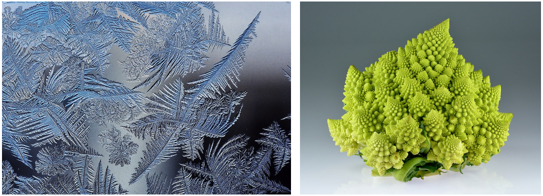
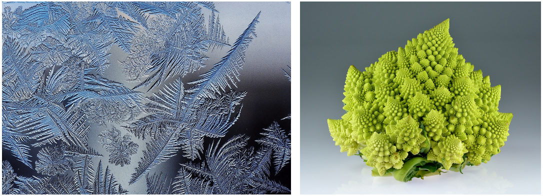


Figure: Natural fractals

- Objective in fractal geometry is to quantify size and complexity of *fractals*

Fractal geometry



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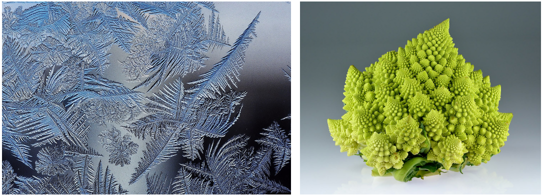
- Fractals are sets with a complicated and detailed structure at arbitrarily small scales

Fractal geometry



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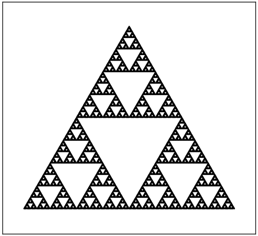
- Objective in fractal geometry is to quantify size and complexity of *fractals*
- Fractals are sets with a complicated and detailed structure at arbitrarily small scales
- Often fractals exhibit a (approximately) self-similar structure

Fractal geometry



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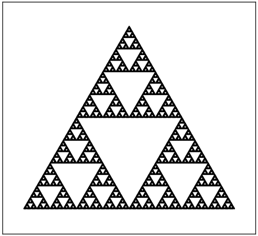
Fractal geometry



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- Most common way to measure size in fractal geometry is via various notions of fractal dimension.

Fractal geometry

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We will study this question for *Assouad dimension* of *self-affine sets*.

Iterated function systems

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A finite collection $\{\varphi_i\}_{i=1}^m$ of contractive self-maps of $\mathbb{R}^d$ is called an iterated function system (IFS).

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Theorem (Hutchinson, 1981)

Every IFS has a unique non-empty and compact set $X \subset \mathbb{R}^d$ satisfying

$$X = \bigcup_{i=1}^{m} \varphi_i(X).$$

This set is called the attractor or the limit set of the IFS.

Examples

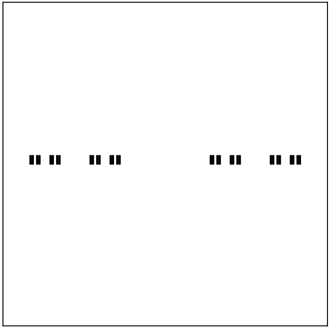


Figure: The Cantor set

Examples

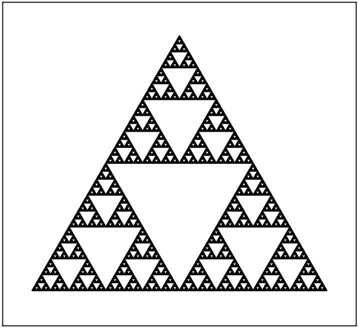


Figure: The Sierpinski triangle

Examples

Figure: The Barnsley fern

Examples

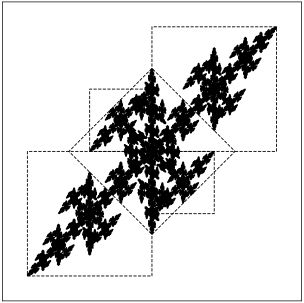


Figure: An overlapping self-similar set

Examples

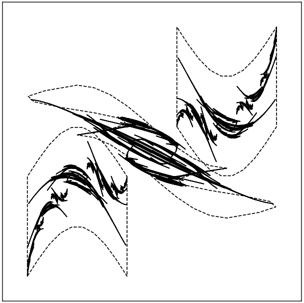
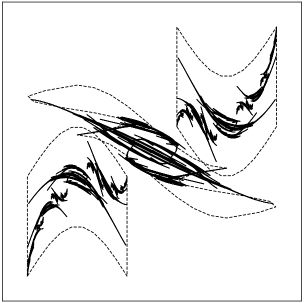


Figure: A non-linear IFS

Iterated function systems



- The general case is extremely difficult to handle.
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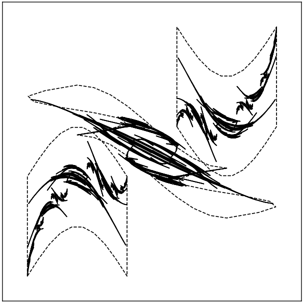
Iterated function systems



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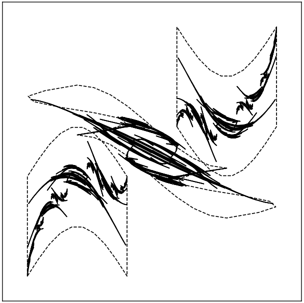
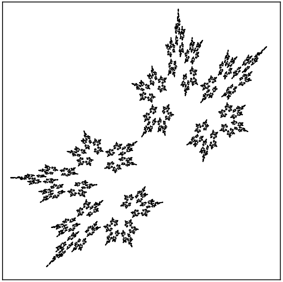


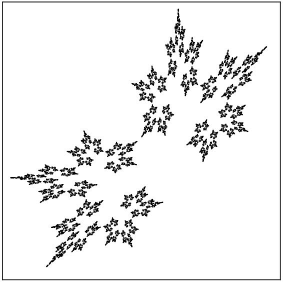
- The general case is extremely difficult to handle.
- To make life easier, on imposes restrictions on
  - (i) The regularity of the maps φ_i in the IFS
  - (ii) The amount of overlap between the images $\varphi_i(X)$.

Self-affine sets



- A finite collection $\{\varphi_i(x) = A_i x + t_i\}_{i=1}^{M}$ of invertible contractive affine self-maps on $\mathbb{R}^2$ is called a self-affine iterated function system (affine IFS).

Self-affine sets



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- In this case the limit set X is called a self-affine set.

Examples

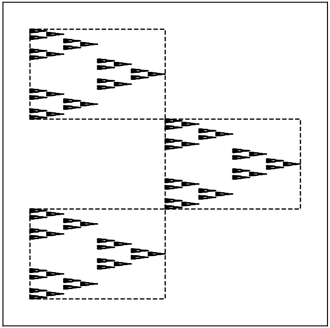

Figure: A Bedford-McMullen carpet

Examples

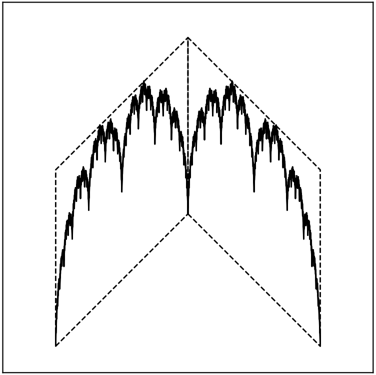

Figure: The Takagi function

Weak tangents

Let $X \subset \mathbb{R}^d$ be compact and $T_{x,r} \colon \mathbb{R}^d \to \mathbb{R}^d$ be a similarity taking $Q(x,r) := x + [0,r]^d$ to the unit cube $Q = [0,1]^d$ in an orientation preserving way.

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$$T_{x_n,r_n}(X) \cap Q \to T$$

in the Hausdorff distance, then T is called a weak tangent of X. The collection of weak tangents of X is denoted by $\mathrm{Tan}(X)$.

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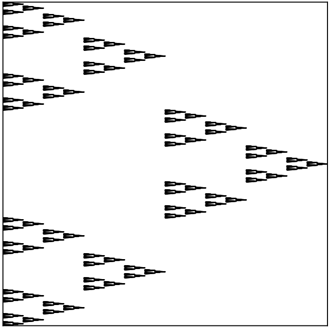
Theorem (Käenmäki-Ojala-Rossi, 2018)

If $X \subset \mathbb{R}^d$ is a compact set, then

$$\dim_{\mathrm{A}}(X) = \max\{\dim_{\mathrm{H}}(T) \colon T \in \mathrm{Tan}(X)\}.$$

Assouad dimension of self-affine sets - Heuristic



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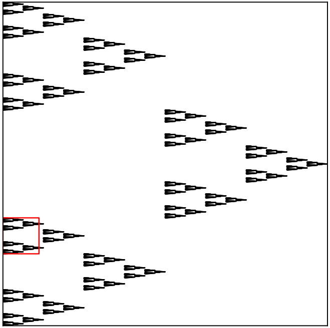

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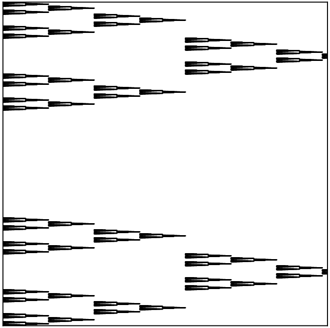

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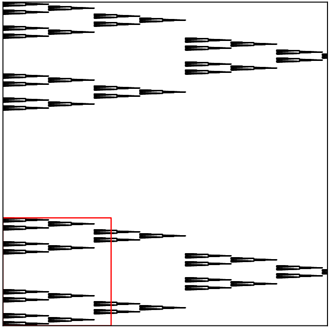

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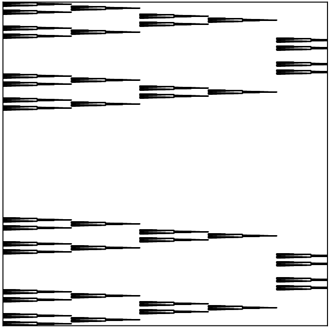

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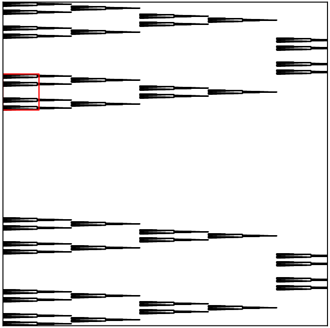

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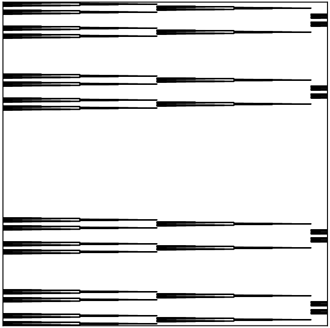

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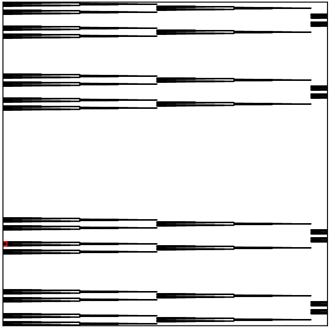

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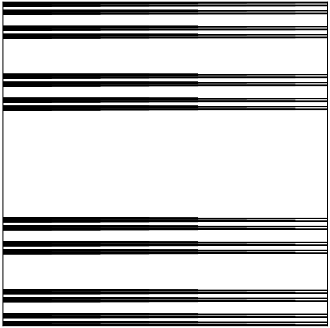

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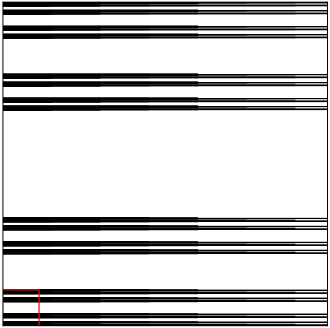

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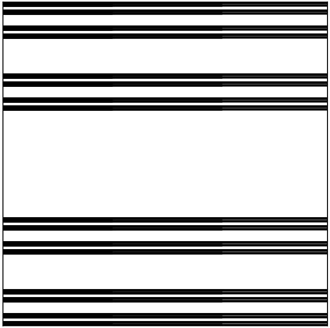

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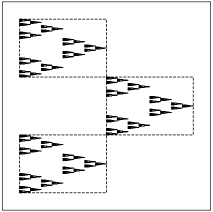


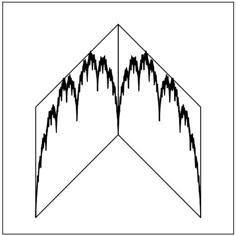
Indeed, the following result was proved by Mackay.

Theorem (Mackay, 2011)

If X is a self-affine carpet with sufficiently nice grid structure which projects to an interval vertically, then

$$\begin{aligned}\dim_A X &= 1 + \max \dim_H(\textit{vertical slice of } X)\\ &= 1 + \max \dim_A(\textit{vertical slice of } X)\end{aligned}$$

Assouad dimension of self-affine sets



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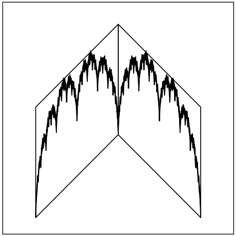

- For the tangents to have a fibre structure, the set needs to have three properties:
  1. Images of the set under long compositions of the maps in the IFS need to get thinner and thinner.

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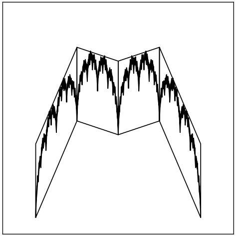

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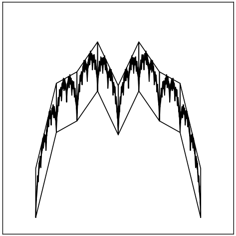

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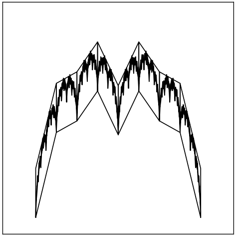

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    2. There are not too severe overlaps.

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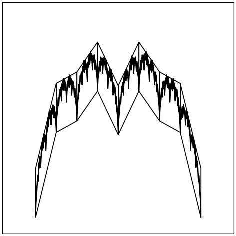

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  3. The set needs to be large enough for the fibres to look like lines.

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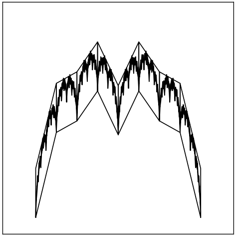

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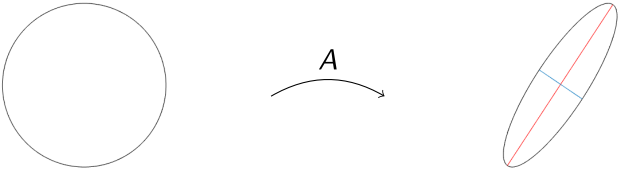
  3. The set needs to be large enough for the fibres to look like lines.
- What are the analogues of vertical and horizontal directions in the general setting?

Definitions



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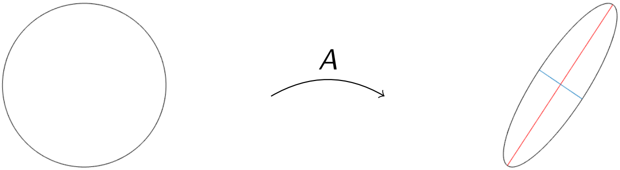
Definitions



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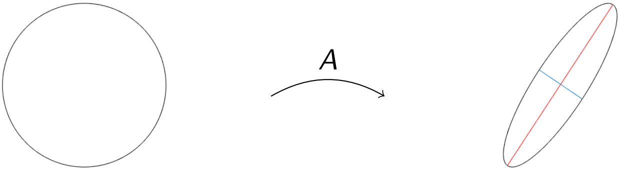

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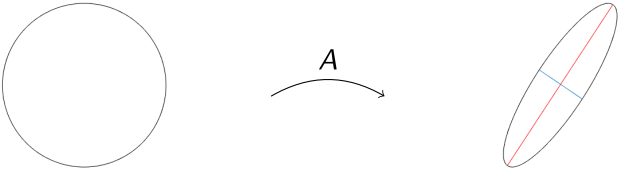

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- Geometrically these are the lengths of the longer and shorter semiaxes of $A(B(0,1))$, respectively.
- We assume strict inequality.
- Let $\vartheta(A)$ denote the line spanned by the longer semiaxis of $A(B(0,1))$.

Domination

A self-affine set X is dominated if there exist constants $C > 0$ and $0 < \tau < 1$, such that

$$\frac{\alpha_2(A_{i_1} \cdot \ldots \cdot A_{i_n})}{\alpha_1(A_{i_1} \cdot \ldots \cdot A_{i_n})} \leqslant C\tau^n,$$

for all $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, M\}$.

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for all $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, M\}$. Let us denote by Y_F the limit directions of $\vartheta(A_{i_1} \cdots A_{i_n})$ and by X_F the limit directions of $\vartheta(A_{i_1}^{-1} \cdots A_{i_n}^{-1})$.

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Lemma

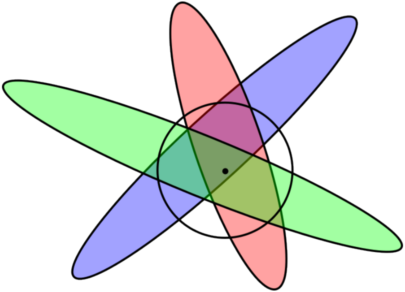
If X is dominated, then the limit directions $\vartheta(A_{i_1} \cdots)$ and $\vartheta(A_{i_1}^{-1} \cdots)$ exist for all sequences and the convergence is uniform. Moreover, the sets Y_F and X_F are disjoint compact sets.

Bounded neighbourhood condition

A self-affine set X satisfies the bounded neighbourhood condition (BNC) if there is a constant M, such that

$$\#\{\varphi_{\mathtt{i}} \mid \alpha_2(A_{\mathtt{i}}) \approx r,\ B(x,r) \cap \varphi_{\mathtt{i}}(X) \neq \emptyset\} \leqslant M,$$

for all $x \in X$ and $r > 0$.



Main result

Theorem (A.-Bárány-Käenmäki, 2023)

If X is a dominated self-affine set satisfying the BNC, such that $\dim_{\mathrm{H}}(\mathrm{proj}_{V^\perp} X) = 1$ for all $V \in X_F$, then

$$\begin{aligned}\dim_{\mathrm{A}}(X) &= 1 + \max_{\substack{x \in X \\ V \in X_F}} \dim_{\mathrm{H}}(X \cap (V + x)) \\ &= 1 + \max_{\substack{x \in X \\ V \in \mathbb{RP}^1 \setminus Y_F}} \dim_{\mathrm{A}}(X \cap (V + x)).\end{aligned}$$

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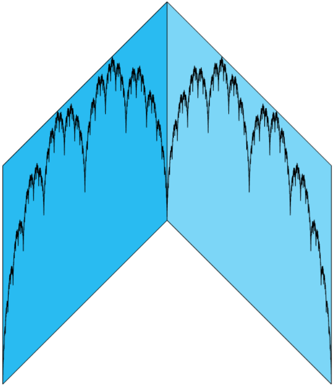
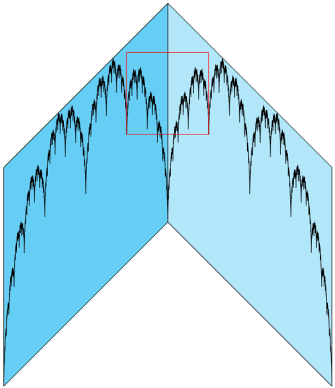
- The projection condition is satisfied if the set has $\dim_{\mathrm{H}} X \geqslant 1$ and the semigroup generated by the linear parts of the affine IFS is strongly irreducible.

Proof



Proof



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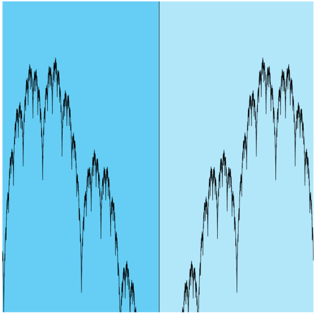

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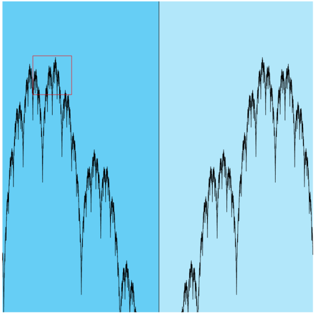

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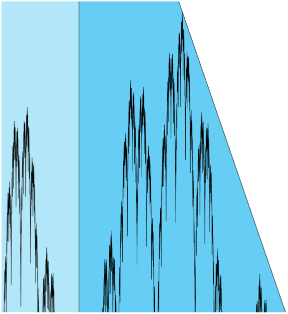

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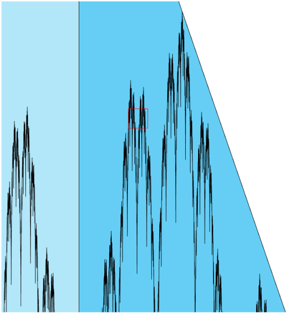

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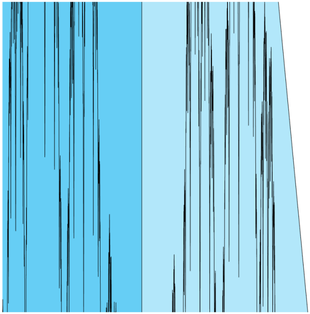

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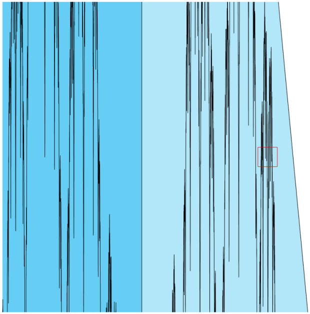

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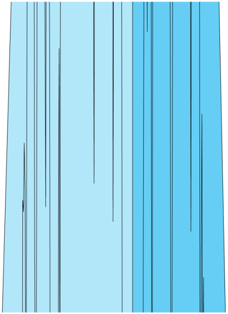

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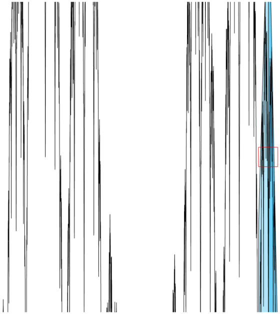

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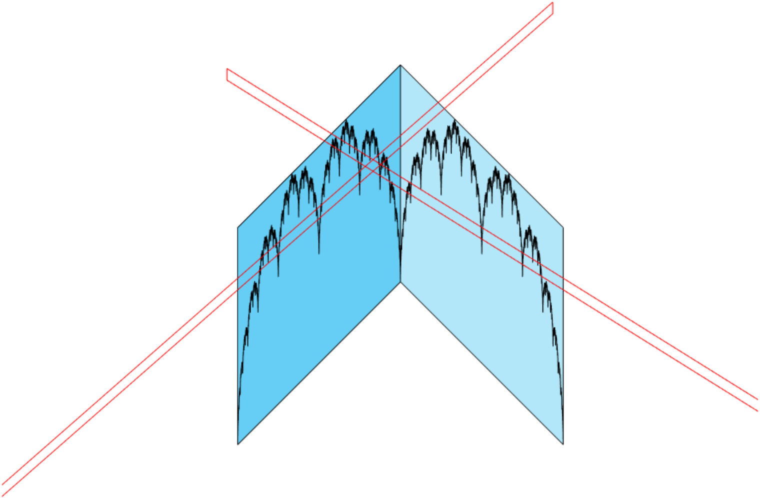

Thank you for your attention!

Questions are welcome!